



## Calhoun: The NPS Institutional Archive

---

Theses and Dissertations

Thesis Collection

---

1998-06

# On the quasimonotonicity of a square linear operator with respect to a nonnegative cone

Beaver, Philip

Monterey, California. Naval Postgraduate School

---

<http://hdl.handle.net/10945/8768>



Calhoun is a project of the Dudley Knox Library at NPS, furthering the precepts and goals of open government and government transparency. All information contained herein has been approved for release by the NPS Public Affairs Officer.

**Dudley Knox Library / Naval Postgraduate School**  
**411 Dyer Road / 1 University Circle**  
**Monterey, California USA 93943**

<http://www.nps.edu/library>



NPS ARCHIVE  
1998.06  
BEAVER, P.



DUDLEY KNOX LIBRARY  
NAVAL POSTGRADUATE SCHOOL  
MONTEREY, CA 93943-5101

DUDLEY KNOX LIBRARY  
NAVAL POSTGRADUATE SCHOOL  
MONTEREY CA 93943-5101



# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



## DISSERTATION

ON THE QUASIMONOTONICITY OF A SQUARE LINEAR  
OPERATOR WITH RESPECT TO A NONNEGATIVE  
CONE

by

Philip Beaver

June, 1998

Dissertation Supervisor:

David Canright

Approved for public release; distribution is unlimited.



# REPORT DOCUMENTATION PAGE

Form Approved  
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instruction, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington DC 20503.

1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE June 1998		3. REPORT TYPE AND DATES COVERED Doctoral Dissertation	
4. TITLE AND SUBTITLE <b>ON THE QUASIMONOTONICITY OF A SQUARE LINEAR OPERATOR WITH RESPECT TO A NONNEGATIVE CONE</b>				5. FUNDING NUMBERS	
6. AUTHOR(S) Beaver, Philip					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey, CA 93943-5000				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSORING / MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.					
12a. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution unlimited.				12b. DISTRIBUTION CODE	
13. <b>ABSTRACT (maximum 200 words)</b> The question of when a square, linear operator is quasimonotone nondecreasing with respect to a nonnegative cone was posed for the application of vector Lyapunov functions in 1974. Necessary conditions were given in 1980, which were based on the spectrum and the first eigenvector. This dissertation gives necessary and sufficient conditions for the case of the real spectrum when the first eigenvector is in the nonnegative orthant, and when the first eigenvector is in the boundary of the nonnegative orthant, it gives conditions based on the reducibility of the matrix. For the complex spectrum, in the presence of a positive first eigenvector the problem is shown to be equivalent to the irreducible nonnegative inverse eigenvalue problem.					
14. SUBJECT TERMS Essentially nonnegative matrices, Vector Lyapunov functions, Nonnegative cones				15. NUMBER OF PAGES 107	
				16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified		20. LIMITATION OF ABSTRACT UL	

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)  
Prescribed by ANSI Std. Z39-18



NPS ARCHIVE

948.06

BEAVER, P.

~~1705~~  
~~83/11/06~~  
~~9.1~~

Approved for public release; distribution is unlimited

**ON THE QUASIMONOTONICITY OF A SQUARE LINEAR OPERATOR WITH  
RESPECT TO A NONNEGATIVE CONE**

Philip Beaver  
Major, United States Army  
B.S., United States Military Academy, 1983  
M.S., Naval Postgraduate School, 1991

Submitted in partial fulfillment of the  
requirements for the degree of

**DOCTOR OF PHILOSOPHY IN APPLIED MATHEMATICS**

from the

**NAVAL POSTGRADUATE SCHOOL**  
**June, 1998**



## ABSTRACT

The question of when a square, linear operator is quasimonotone nondecreasing with respect to a nonnegative cone was posed in 1974 for the application of vector Lyapunov functions. Necessary conditions were given in 1980 based on the spectrum and the first eigenvector. This dissertation gives necessary and sufficient conditions for the case of the real spectrum when the first eigenvector is positive, and when the first eigenvector is nonnegative it gives conditions based on the reducibility of the matrix. For the complex spectrum, in the presence of a positive first eigenvector the problem is shown to be equivalent to the irreducible nonnegative inverse eigenvalue problem.





## TABLE OF CONTENTS

I. INTRODUCTION.....	1
II. CONES AND NONNEGATIVE MATRICES.....	5
A. CONES IN EUCLIDEAN SPACE.....	5
B. NONNEGATIVE MATRICES.....	9
III. CONE-VALUED VECTOR LYAPUNOV FUNCTIONS.....	13
A. DIFFERENTIAL EQUATIONS, INEQUALITIES, AND STABILITY.....	13
B. CONE-VALUED VECTOR LYAPUNOV FUNCTIONS.....	30
C. THE QUASIMONOTONICITY OF LINEAR DIFFERENTIAL SYSTEMS...	36
D. STABILITY THROUGH LINEAR COMPARISON SYSTEMS.....	40
IV. THE QUASIMONOTONICITY OF A SQUARE, LINEAR OPERATOR WITH RESPECT TO A NONNEGATIVE CONE: THE REAL SPECTRUM.....	53
A. MATRICES WITH A POSITIVE FIRST EIGENVECTOR.....	53
B. REDUCIBLE MATRICES WITH A NONNEGATIVE FIRST EIGENVECTOR.....	57
C. IRREDUCIBLE MATRICES WITH A NONNEGATIVE FIRST EIGENVECTOR.....	62
V. THE QUASIMONOTONICITY OF A SQUARE, LINEAR OPERATOR WITH RESPECT TO A NONNEGATIVE CONE: THE GENERAL SPECTRUM.....	65
A. IRREDUCIBLE, ESSENTIALLY NONNEGATIVE MATRICES.....	65
B. THE NONNEGATIVE INVERSE EIGENVALUE PROBLEM.....	69
VI. FURTHER APPLICATIONS.....	73
VII. CONCLUSIONS AND FUTURE RESEARCH.....	81
LIST OF REFERENCES.....	85
INITIAL DISTRIBUTION LIST.....	95



## ACKNOWLEDGMENT

I first want to thank my advisor, David Canright, for his patience, guidance, and careful review of my work during the conduct of this research. His exacting standards not only influenced this document, but also will effect all of my future work. He deserves the free time he will now have.

My committee not only helped me through my research and the preparation of this dissertation, but they were invaluable in getting me through the entire program. For this I owe a debt of gratitude to Chris Frenzen (Chairman), Rob Dell, Clyde Scandrett, and Maury Weir.

I owe a special thanks to Don Fausett, who introduced me to this problem, who got me started on my research, and who continued to help me through the entire process.

I received a lot of guidance from other faculty members, who all took the time to offer suggestions, critique my ideas, and discuss applications. I want to thank Van Henson from Lawrence Livermore National Laboratory; Bill Gragg, Wei Kang, Carlos Borges, Guillermo Owen, and Gordon Latta of the Naval Postgraduate School Department of Mathematics; and Marvin Marcus from the University of California at Santa Barbara.

Finally, I received a lot of moral support and encouragement from my family, who experienced every up and down with me for the last three years. For this I thank my parents; my girls, Elaine and Elizabeth; and especially my wife, Kim.

While I received a lot of help and guidance in my research, I retain sole responsibility for any errors this dissertation may contain.





# I. INTRODUCTION

This dissertation addresses what appears to be a simple problem: “Given a real, square matrix  $A$ , when does there exist a nonnegative matrix  $B$  such that  $B^{-1}AB$  has its off-diagonal elements nonnegative?” Based on different bodies of literature, and on the application, we could similarly ask when a square linear operator is quasimonotone nondecreasing with respect to a cone in the nonnegative orthant, or when a real matrix is essentially nonnegative under a nonnegative change of basis.

This question has its roots in the Perron-Frobenius theory of the early 1900s, and as we develop our solution, we trace its history, as well as the history and current state of some of its applications. We also discuss similar and related problems to demonstrate how this problem fits into the scheme of applied mathematics.

We present a brief background on cones in Euclidean space and on the theory of nonnegative matrices in Chapter II. We use the following convention to present results. Theorems (and proofs) from the original source have the author and date after the theorem number. If the proof comes from another source, we indicate that prior to presenting the theorem. For well-known results, we either indicate the source or provide our own proof. We indicate the completion of a theorem and proof (or just a theorem for results presented without proof) with the  $\square$  symbol.

Chapter III motivates this question as an unsolved problem from the theory of cone-valued vector Lyapunov functions, which can determine stability in dynamical systems. We also present some of our results concerning this application. This question

was first posed for the application of vector Lyapunov functions in 1974, and partial answers were given for this setting in 1980 and 1995.

Chapters IV and V present our original solutions to this problem. The problem has significantly different solutions depending on whether the spectrum of  $A$  is real or general, and Chapter IV presents our solution for the real spectrum. In Section IV.A we show the necessary condition that the first eigenvector  $x_1$  of  $A$  (associated with the greatest eigenvalue  $\lambda_1$ ) be nonnegative, and the sufficient condition that  $x_1$  be positive.

The case where  $x_1$  is nonnegative but not strictly positive has different solutions based on the reducibility of the matrix  $A$ . Section IV.B addresses the reduced case, where we show sufficient conditions when  $\lambda_1$  is a simple eigenvalue of  $A$ , and we have a sufficient (but not necessary) condition when  $\lambda_1$  is not simple.

In the case where  $x_1$  is nonnegative and  $A$  is irreducible, Section IV.C shows that we can use  $x_1$  to deflate the matrix  $A$  and reduce the problem to one of dimension  $n - 1$ . Here, we also give necessary and sufficient conditions for a cone to exist.

Chapter V presents our solution for the complex spectrum. We show that in the presence of a positive first eigenvector, the problem can be reduced to the nonnegative inverse eigenvalue problem, a classic unsolved problem from theoretical linear algebra. We present the solution only in terms of irreducible quasimonotone matrices, and we show why it is sufficient to consider such matrices.

Chapter VI presents other direct and related applications from control theory and dynamics, as well as from other fields of applied mathematics, and we mention similar problems which have been solved or discussed previously from these and other fields.

Chapter VII summarizes our results, to include the unsolved problems mentioned above, and we suggest further research.

One attractive aspect of our solutions is they are constructive, and hence fairly simple, although there is some numerical instability inherent in some constructions. Particularly for applications where the existence of a matrix  $B$  is sufficient information for the analyst, we present collections of useful necessary and sufficient conditions. We begin with some background.





## II. CONES AND NONNEGATIVE MATRICES

This chapter provides background for the main results. Specifically, we discuss cones in Euclidean space and present results from the theory of nonnegative matrices. We begin with a discussion of cones.

### A. CONES IN EUCLIDEAN SPACE.

The employment of cones as subsets of  $\mathbf{R}^n$  began in the 1930s, where along with convex polyhedra, they were soon found to be useful in the field of linear optimization and in the study of linear inequalities (see, for example, Weyl, 1935; and Goldman and Tucker, 1956). Dual cones were first employed in 1941 by J. Dieudonné (see Lay, 1982), although dual sets to convex sets had been used much earlier (see Helly, 1923).

The analysis of cones, convex polyhedra, and convex sets in general has become a major field of study in its own right, which has its roots in Caratheodory (1907), and which can be found, for example, in Sandgren (1954), Karlin (1968), Rockafeller (1970), Stoer and Witzgall (1970), Berman (1972), and Lay (1982). We need only the most basic definitions and results from those fields, and we begin with some definitions.

If  $\mathbf{R}^n$  is Euclidean  $n$ -space with norm  $\|\cdot\|$  and inner product  $\langle \cdot, \cdot \rangle$ , a subset  $S \subseteq \mathbf{R}^n$  is *convex* if and only if  $x, y \in S$  implies  $\alpha x + (1 - \alpha)y \in S$  for all  $\alpha \in [0, 1]$ . A set  $K$  is a *cone* if and only if  $\lambda K \subseteq K$  for  $\lambda \geq 0$ . A *convex cone* satisfies both definitions, and we require that  $K = \overline{K}$ , where  $\overline{K}$  is the closure of  $K$ , so that our cones are *closed*. (It is a result of Farkas (1901) that a set is a closed, convex cone if and only if  $K = (K^*)^*$ , where  $K^*$  is the dual of  $K$ , to be defined shortly.) A cone is *pointed* if and

only if  $K \cap (-K) = \{0\}$ , and *solid* if and only if  $K^\circ$  is nonempty, where  $K^\circ$  is the interior of  $K$ . An equivalent notion (in Euclidean space) to solid is *reproducing*, where  $K$  is reproducing if and only if  $K - K = \mathbb{R}^n$ . (It is a result of Krein and Rutman (1948) that a cone is pointed if and only if its dual is solid.)

A convex cone  $K$  is *polyhedral* if and only if it is generated by finitely many vectors, and if the number of independent vectors is equal to  $n$ , then  $K$  is *simplicial*. A closed, pointed convex cone is called *proper*. The result that a proper cone is generated by its extremals is a special case of the Krein-Milman Theorem. The only cones we consider for our applications are proper, simplicial cones.

Since a proper, simplicial cone is generated by  $n$  independent extremal vectors  $b_i$ , we consider the nonsingular matrix  $B = [b_1, \dots, b_n] \in \mathbb{R}^{n \times n}$ , and we denote the cone generated by the columns of  $B$  as  $K(B)$ . Clearly,  $K(B) = \{x \in \mathbb{R}^n | x = \sum_{i=1}^n \omega_i b_i, \omega_i \geq 0\}$ .

The cone we most frequently encounter is  $\mathbb{R}_+^n$ , the *nonnegative orthant*, where

$$\mathbb{R}_+^n = K(I) \text{ with } I \text{ being the identity matrix in } \mathbb{R}^{n \times n}.$$

The cone  $K$  induces an order relation on  $\mathbb{R}^n$  by  $x \leq_K y \Leftrightarrow y - x \in K$ , and  $x <_K y \Leftrightarrow y - x \in K^\circ$ . The *dual*, or adjoint, cone is  $K^* = \{\phi \in \mathbb{R}^n | \langle \phi, x \rangle \geq 0 \forall x \in K\}$ , which clearly satisfies the properties of a cone, and for a proper, simplicial cone,  $(K^*)^* = K$ . If we define  $K_0$  as  $K \setminus \{0\}$ , then  $x \in K^\circ \Leftrightarrow \langle \phi, x \rangle > 0 \forall \phi \in K_0^*$ , and  $x \in \partial K \Leftrightarrow \langle \phi, x \rangle = 0$  for some  $\phi \in K_0^*$ , where  $\partial K$  is the boundary of  $K$ .

A continuous function  $f: D \subseteq \mathbf{R}^n \rightarrow \mathbf{R}^n$  is *quasimonotone nondecreasing* in  $x$  relative to the cone  $K$  if for  $x, y \in D$ ,  $y - x \in \partial K$  implies there exists a  $\phi \in K_0^*$  such that  $\langle \phi, y - x \rangle = 0$  and  $\langle \phi, f(y) - f(x) \rangle \geq 0$ . This definition, from Elsner (1974), is fairly standard; however, for nonlinear functions some authors require  $x, y \in K$  (see, for example, Heikkilä, 1983). For a linear function  $f(x) = Ax$ ,  $A \in \mathbf{R}^{n \times n}$ , the quasimonotone nondecreasing property reduces to:  $x \in \partial K$  implies there exists a  $\phi \in K_0^*$  such that  $\langle \phi, x \rangle = 0$  and  $\langle \phi, Ax \rangle \geq 0$ .

An equivalent definition is  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is quasimonotone nondecreasing if and only if  $f_i(x)$  is nondecreasing in  $x_j$  for all  $i, j = 1, \dots, n$ ,  $i \neq j$ , so the following lemma characterizes quasimonotone nondecreasing linear operators. This result appears as an example in Lakshmikantham and Leela (1977b).

**Lemma A.1.** A linear operator  $A \in \mathbf{R}^{n \times n}$  is quasimonotone nondecreasing relative to the nonnegative orthant if  $a_{ij} \geq 0$  for all  $i \neq j$ .

**Proof.** We prove this for  $n = 2$ . A similar argument works for  $n > 2$ . Let  $K = \mathbf{R}_+^2 = \{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0\}$ . Then  $K^* = K$ , since  $\langle \phi, x \rangle \geq 0$  implies  $\phi_1 x_1 + \phi_2 x_2 \geq 0$  for all  $x_1, x_2 \geq 0$ , which in turn implies  $\phi_1, \phi_2 \geq 0$ . If  $f(x) = Ax$ , where

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  is quasimonotone nondecreasing relative to  $K$ , then for  $x_1 = 0$  or

$x_2 = 0$ , there exists a  $\phi \in K_0^*$  such that  $\langle \phi, x \rangle = 0$  and  $\langle \phi, Ax \rangle \geq 0$ . Now

$\langle \phi, x \rangle = \phi_1 x_1 + \phi_2 x_2 = 0$  and  $\langle \phi, Ax \rangle = \phi_1(a_{11}x_1 + a_{12}x_2) + \phi_2(a_{21}x_1 + a_{22}x_2) \geq 0$ . If  $x_1 = 0$ ,

$x_2 \neq 0$ , then  $\phi_2 x_2 = 0$  so  $\phi_2 = 0$ . This implies, since  $\phi_1 > 0$ , that  $\phi_1 a_{12} x_2 \geq 0$ , so  $a_{12} \geq 0$ .



Similarly,  $a_{21} \geq 0$ , so the linear functions which are quasimonotone nondecreasing relative to  $K$  are precisely those matrices with nonnegative off-diagonal entries.  $\square$

We use the following well-known results.

**Lemma A.2.** Given a nonsingular matrix  $B \in \mathbf{R}^n$  and the cone  $K(B)$ , then  $K((B^{-1})^T) = K(B)^*$ .

**Proof.** Let  $x \in K((B^{-1})^T)$  and  $y \in K(B)$  so that  $x = \sum_{i=1}^n \alpha_i \phi_i$  and  $y = \sum_{i=1}^n \beta_i b_i$ ,

with  $\alpha_i, \beta_i \geq 0$ . Then  $\langle x, y \rangle = \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} \alpha_i \beta_j = \sum_{i=1}^n \alpha_i \beta_i \geq 0$ , so  $K((B^{-1})^T) \subseteq K(B)^*$ .

Similarly, let  $x \notin K((B^{-1})^T)$ . Since the  $\phi_i$  are independent,  $x = \sum_{i=1}^n \gamma_i \phi_i$ , but there

exists at least one  $\gamma_j < 0$ . For  $y = b_j \in K(B)$ , it follows that  $\langle x, y \rangle = \gamma_j < 0$ , so  $x \notin K(B)^*$ .

Hence,  $K(B)^* \subseteq K((B^{-1})^T)$ , proving the lemma.  $\square$

**Lemma A.3.** Given a nonsingular matrix  $B \in \mathbf{R}^n$  and the cone  $K(B)$ , then

$K(B) \subseteq \mathbf{R}_+^n$  if and only if  $K(B^{-1}) \supseteq \mathbf{R}_+^n$ .

**Proof.** Let  $K(B) \subseteq \mathbf{R}_+^n$  and let  $a \in \mathbf{R}_+^n$  be arbitrary. Then  $Ba = x \in \mathbf{R}_+^n$  since  $x$  is a positive linear combination of the generators of  $K(B)$ . Because  $B$  is invertible, it follows that  $a = B^{-1}x$ . As  $a \in \mathbf{R}_+^n$  is arbitrary and expressible as a nonnegative linear combination of the columns of  $B^{-1}$ , we conclude that  $K(B^{-1}) \supseteq \mathbf{R}_+^n$ .

Conversely, assume  $K(B^{-1}) \supseteq \mathbf{R}_+^n$  and let  $a \in \mathbf{R}_+^n$ . Then there exists an

$x \in \mathbf{R}_+^n$  such that  $B^{-1}x = a$ , so  $x = Ba$ . Then  $B$  is nonnegative, otherwise it has some column  $b_j$  which is not nonnegative, and letting  $a = [0_1, \dots, 0_{j-1}, 1_j, 0_{j+1}, \dots, 0_n]^T$  shows  $x \notin \mathbf{R}_+^n$ , which is a contradiction. This proves the lemma.  $\square$

Since  $K(B) \subseteq \mathbf{R}_+^n$  implies  $B$  is a nonnegative matrix, and since  $B^T$  is also nonnegative, then  $K((B^{-1})^T) \supseteq \mathbf{R}_+^n$  as well. This gives a nice characterization of the inverses of nonnegative matrices.

When we say that a matrix  $A \in \mathbf{R}^{n \times n}$  (or a linear operator  $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ ) is quasimonotone nondecreasing with respect to a cone  $K(B)$ , we mean that  $C = B^{-1}AB$  is quasimonotone nondecreasing (with respect to  $\mathbf{R}_+^n$ ) even though  $A$  may not be. In Chapter III we show this result as a theorem of Heikkilä.

Since quasimonotonicity is related to nonnegative matrices, we next present some background on nonnegative matrices.

## B. NONNEGATIVE MATRICES

The study of nonnegative matrices began with Perron's (1907) presentation of a theory of positive matrices. Frobenius (1908, 1909, 1912) immediately extended this theory to nonnegative matrices, and the general theory has carried their names ever since. Here we present only their basic results required for our application.

A *positive matrix*  $A \in \mathbf{R}^{n \times n}$  has  $a_{ij} > 0$  for  $i, j = 1, \dots, n$ . A *nonnegative matrix* has  $a_{ij} \geq 0$ . A matrix  $A$  is *essentially nonnegative (positive)* if and only if  $a_{ij} \geq 0$  ( $> 0$ ) for all  $i \neq j$ . The following lemma justifies the term “essentially” nonnegative.

**Lemma B.1.** If a matrix  $A \in \mathbf{R}^{n \times n}$  is essentially nonnegative, then there exists a number  $r > 0$  such that  $\hat{A} = A + rI$  is nonnegative. Furthermore, the spectrum  $\sigma(\hat{A}) = \sigma(A) + r$ , and the eigenvectors of  $A$  and  $\hat{A}$  are equal.

**Proof.** The first statement is obvious. Let  $Ax = \lambda x$ . Then  $\hat{A}x = (A + rI)x = Ax + rIx = \lambda x + rx = (\lambda + r)x$ , completing the proof.  $\square$

Hence, we can shift the diagonal (and the spectrum) of an essentially nonnegative (positive) matrix to make it nonnegative (positive).

A matrix  $A$  is *reducible* if it is permutation similar to a matrix

$$A_p = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \text{ where } A_{11} \text{ and } A_{22} \text{ are square (and of positive dimension).}$$

Otherwise,  $A$  is *irreducible*. A positive matrix is a special case of a nonnegative, irreducible matrix. The following theorem summarizes the basic results we need from the theory of nonnegative matrices.

**Theorem B.2.** (Perron, 1907; Frobenius, 1912). A positive matrix  $A > 0$  has an eigenvalue equal to the spectral radius of  $A$ . Associated with this eigenvalue is a positive eigenvector. Furthermore,  $A$  has no other nonnegative eigenvectors. The same is true for a nonnegative, irreducible matrix  $A \geq 0$ .

A nonnegative matrix  $A \geq 0$  has a (real) eigenvalue equal to the spectral radius of  $A$ . Associated with this eigenvalue is a nonnegative eigenvector.  $\square$

For a proof of the above results see, for example, Gantmacher (1959), Varga (1962), or Horn and Johnson (1991). Since the transpose of a matrix with any of the above properties retains that particular property, the same conclusion can be made about

the left eigenvectors of  $A$ . Note that when comparing two matrices as in  $A > B$ , or two vectors as in  $x > y$ , we refer to component-wise majorization.

From Lemma B.1 and Theorem B.2 it follows that an essentially positive matrix has a real eigenvalue with greatest real part, with positive left and right eigenvectors associated with this eigenvalue, and no other nonnegative eigenvectors. Similar conclusions follow for essentially nonnegative and irreducible essentially nonnegative matrices.

If a linear operator  $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is quasimonotone nondecreasing, then the matrix  $A \in \mathbf{R}^{n \times n}$  is essentially nonnegative. Therefore, if we ask when a square, linear operator  $A$  on  $\mathbf{R}^n$  is quasimonotone nondecreasing with respect to a cone in  $\mathbf{R}_+^n$ , we are asking the equivalent question of when the matrix  $A \in \mathbf{R}^{n \times n}$  is essentially nonnegative under a nonnegative change of basis. The next chapter introduces the motivating application, the theory of cone-valued vector Lyapunov functions.

1. The first part of the paper discusses the importance of the research and the objectives of the study. It also provides a brief overview of the methodology used in the study.

2. The second part of the paper presents the results of the study. It includes a detailed description of the data collected and the analysis performed. The results are presented in a clear and concise manner, with appropriate use of tables and figures.

3. The third part of the paper discusses the implications of the findings. It explores the potential applications of the research and the limitations of the study. It also provides a brief overview of the conclusions drawn from the study.

4. The fourth part of the paper provides a summary of the findings and a brief overview of the conclusions drawn from the study. It also includes a brief discussion of the limitations of the study and the potential for future research.

5. The fifth part of the paper provides a brief overview of the conclusions drawn from the study. It also includes a brief discussion of the limitations of the study and the potential for future research.

6. The sixth part of the paper provides a brief overview of the conclusions drawn from the study. It also includes a brief discussion of the limitations of the study and the potential for future research.

7. The seventh part of the paper provides a brief overview of the conclusions drawn from the study. It also includes a brief discussion of the limitations of the study and the potential for future research.

8. The eighth part of the paper provides a brief overview of the conclusions drawn from the study. It also includes a brief discussion of the limitations of the study and the potential for future research.



### III. CONE-VALUED VECTOR LYAPUNOV FUNCTIONS

The motivating problem for determining the quasimonotonicity of a square matrix with respect to a nonnegative cone in  $\mathbf{R}^n$  comes from stability theory in differential equations. Specifically, we seek such a cone when using vector Lyapunov functions to determine the stability of an equilibrium in a dynamical system. This chapter presents this problem for linear comparison systems.

We begin with a brief introduction to the theory of differential equations and stability. In order to develop the technique of vector Lyapunov functions, we also require results from the theory of differential inequalities. We present only the results from these fields needed to prove the major theorems of vector Lyapunov functions. In the case of well-known results, or those which will not be used directly, we present the results without proof.

#### A. DIFFERENTIAL EQUATIONS , INEQUALITIES, AND STABILITY

We consider a system of first-order differential equations

$$x' = f(t, x), \quad x(t_0) = x_0, \quad (\text{A.1})$$

where  $x' \equiv dx/dt$ ,  $x \in \mathbf{R}^m$ ,  $f: I \times D \rightarrow \mathbf{R}^m$  where  $I \subset \mathbf{R}$  and  $D \subset \mathbf{R}^m$  are open sets containing  $t_0$  and  $x_0$  respectively, and  $f$  has certain continuity requirements. A *solution*  $x(t)$  to equation A.1 is a differentiable function  $x: J \subset \mathbf{R} \rightarrow \mathbf{R}^m$  satisfying the differential equation  $dx/dt = f(t, x)$  and the *initial condition*  $x(t_0) = x_0$ . We also consider the *autonomous* system

$$x' = f(x), \quad x(t_0) = x_0, \quad (\text{A.2})$$

where the vector field is independent of time.

Picard (1890) and Lindelöf (1894) showed that if  $f(t, x)$  is continuous on a closed set containing  $(t_0, x_0)$ , and is uniformly Lipschitz continuous with respect to  $x$  on this set, then equation A.1 has a unique solution  $x(t)$  on some set  $[t_0, t_0 + \alpha]$  for  $\alpha > 0$ . Peano (1890) showed that without the assumption of Lipschitz continuity, then equation A.1 has at least one solution  $x(t)$  on  $t \in [t_0, t_0 + \alpha]$ . Furthermore, a solution on an open subset of  $\mathbf{R}^{m+1}$  containing  $(t_0, x_0)$  can be extended to the boundary of that set. Proofs of these results can be found, for example, in Hartman (1982), and these existence and uniqueness results are the basis for the theory of differential equations.

To prove stability results about equilibria in equation A.1, we also need results from the theory of differential inequalities. We extend the concept of the standard derivative used in equation A.1 to the Dini derivative, defined as  $D_-x(t) = \liminf_{h \rightarrow 0^+} (1/h)[x(t+h) - x(t)]$ . This is one of four Dini derivatives, and we define the others as needed. We begin with scalar inequalities ( $m = 1$  in equation A.1). The following results are found, for example, in Lakshmikantham and Leela (1969a) or Walter (1970).

**Theorem A.1.** Let  $E$  be an open  $(t, x)$ -set in  $\mathbf{R}^2$  and  $f \in C[E, \mathbf{R}]$ . Assume that  $v, w \in C[[t_0, t_0 + \alpha), \mathbf{R}]$  for some  $\alpha > 0$ , and  $(t, v(t)), (t, w(t)) \in E$  for  $t \in [t_0, t_0 + \alpha)$ . If  $v(t_0) < w(t_0)$  and if  $D_-v(t) \leq f(t, v(t))$  and  $D_-w(t) > f(t, w(t))$  for  $t \in (t_0, t_0 + \alpha)$ , then for  $t \in [t_0, t_0 + \alpha)$ ,  $v(t) < w(t)$ .  $\square$

For the scalar differential equation A.1 ( $m = 1$ ), we define the *maximal solution* on  $[t_0, t_0 + \alpha)$  as the unique  $r(t)$  such that for every solution  $x(t)$  defined on  $[t_0, t_0 + \alpha)$ ,

$x(t) \leq r(t)$  for  $t \in [t_0, t_0 + \alpha]$ . The *minimal solution* is defined analogously, and the following results presented for maximal solutions are also valid for minimal solutions, with appropriate changes. Under the hypothesis of Peano's theorem, a maximal solution exists on the interval  $[t_0, t_0 + \alpha]$ , and as before, if one exists on an open set, it can also be extended to the boundary of that set.

It is useful when employing vector Lyapunov functions to compare a solution to a maximal solution over an interval. The following comparison theorem is basic to this idea.

**Theorem A.2.** Let  $f \in C[E, \mathbf{R}]$ , where  $E$  is an open  $(t, x)$ -set in  $\mathbf{R}^2$ , and let  $[t_0, t_0 + \alpha]$  be the largest interval in which the maximal solution  $r(t)$  of equation A.1 exists. Let  $p \in C[(t_0, t_0 + \alpha), \mathbf{R}]$ ,  $(t, p(t)) \in E$  for  $t \in [t_0, t_0 + \alpha]$ ,  $p(t_0) \leq x_0$ , and for a fixed Dini derivative,  $Dp(t) \leq f(t, p(t))$  on  $t \in [t_0, t_0 + \alpha]$  except on possibly a countable subset of this interval. Then for  $t \in [t_0, t_0 + \alpha]$ ,  $p(t) \leq r(t)$ .  $\square$

The question of the existence of a maximal solution for a system of differential inequalities was solved by Wazewski (1950). He extended the above result to the case  $m > 1$  in equation A.1, and showed that a sufficient condition for a maximal solution to exist to equation A.1 is that  $f_i(x)$  be nondecreasing in  $x_j$  for each  $t \in [t_0, t_0 + \alpha]$ , a property which we defined in Chapter II as quasimonotone nondecreasing. This property was first recognized as being important to differential inequalities by Müller (1926) (see Walter, 1970). There are analogous definitions for increasing, decreasing, etc., as well as for mixed quasimonotone properties.

Burton and Whyburn (1952) used the mixed quasimonotone property to prove the existence of what are known as minimax solutions to differential equations. These are

solutions which, for example, are minimal in the first  $k$  components and maximal in the last  $m - k$  components. The existence of minimax solutions leads to a family of comparison theorems for differential inequalities which can be used to prove results about stability of equilibria using vector Lyapunov functions. These include results about stability, instability, and conditional stability. We present only the comparison theorem for stability, since we use stability for the motivating example of cone-valued vector Lyapunov functions.

**Theorem A.3.** Let  $E$  be an open  $(t, x)$ -set in  $\mathbf{R}^{m+1}$ , with  $f \in C[E, \mathbf{R}^m]$ .

Suppose that  $f$  is quasimonotone nondecreasing in  $x$ , and let  $[t_0, t_0 + \alpha)$  be the largest interval of existence of the maximal solution  $r(t)$  of equation A.1. Let

$p \in C[[t_0, t_0 + \alpha), \mathbf{R}^m]$ ,  $(t, p(t)) \in E$  for  $t \in [t_0, t_0 + \alpha)$ , and on this interval, for the Dini derivative,  $D_-p(t) \leq f(t, p(t))$ . Then  $p(t_0) \leq x_0$  implies  $p(t) \leq r(t)$  on this interval.  $\square$

This theorem applies for any Dini Derivative. The proof of this result, along with all the above results on differential inequalities, are in Lakshmikantham and Leela (1969a).

We now present some basic definitions and concepts from stability theory which allow us to develop the technique of vector Lyapunov functions in its most simple setting.

An *equilibrium* solution to equation A.1 is a point  $\xi$  such that  $f(t, \xi) = 0$  for all  $t$ . Equation A.1 admits the *trivial solution*  $\xi = 0$  if  $f(t, 0) = 0$  for all  $t$ , and we frequently use this as our equilibrium as we can generally send an equilibrium point to the trivial solution via a change of coordinates.

We wish to determine the stability of the equilibria of equation A.1, and we begin with some basic definitions of stability. The trivial solution of equation A.1 is *stable* if for

every neighborhood  $U_1$  of the origin and every  $t_0 \geq 0$ , there is a neighborhood  $U_2$  of the origin such that  $x_0 \in U_2$  implies  $x(t) \in U_1$  for all  $t \geq t_0$ , where  $x(t)$  is a solution satisfying  $x(t_0) = x_0$ . The trivial solution is *asymptotically stable* if it is stable and there exists a neighborhood  $U_3$  of the origin such that  $x_0 \in U_3$  implies that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

We may express these definitions in terms of norms as follows. The trivial solution is stable if the solution  $x(t)$  with  $x(t_0) = x_0$  is such that for every  $\varepsilon > 0$ ,  $t_0 \geq 0$ , there exists a  $\delta > 0$  such that  $\|x_0\| < \delta$  implies  $\|x(t)\| < \varepsilon$  for all  $t \geq t_0$ . Similarly, the trivial solution is asymptotically stable if there exists a  $\delta > 0$  such that  $\|x_0\| < \delta$  implies that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

The trivial solution in a linear system (when  $f(t, x)$  in equation A.1 is a linear function of  $x$ , i.e.,  $f(t, x) = A_t(x)$  where  $A \in \mathbf{R}^{m \times m}$ ) is asymptotically stable when the eigenvalues of the matrix  $A$  all have negative real part for  $t \geq t_0$ . In this case we refer to the origin as a *sink*. In a nonlinear system, if the eigenvalues of the linearized system in a neighborhood of the equilibrium all have negative real part, then that equilibrium is a sink.

If the origin is a sink in a linear system, then for all solutions  $x(t)$ ,  $\|x(t)\|$  is a strictly decreasing function of  $t$ , where  $\|\cdot\|$  is the Euclidean norm. Similarly, it is a well-known result (see, e.g., Hirsch and Smale, 1974; or Arnold, 1989) that when the origin is asymptotically stable in a nonlinear system, there exists some norm such that  $\|x(t)\|$  is a decreasing function of  $t$  for all solutions  $x(t)$  starting sufficiently near the origin.

These basic definitions of stability have numerous refinements. The two major classes of stability definitions are Lyapunov and Poisson, and among the logical statements



allowed in the definitions are 17,017,969 possible types of stability (see Bushaw, 1969). Many of these conditions are meaningless, and Massera (1949, 1956) discusses those which are most often studied in terms of Lyapunov functions. Habets and Peiffer (1973) extend these to a classification which encompasses all possible types of stability.

The above definitions of stability can be generalized for nonautonomous systems (for example, when  $\delta$  depends on  $t_0$ ) and they can be extended to conditional stability (instability) definitions for stable (unstable) manifolds through the point  $(t_0, 0)$ . There is a well-developed theory of vector Lyapunov functions for conditional stability which uses the minimax solutions of Burton and Whyburn, but we discuss only the most basic case. Furthermore, these stability results are local, and we do not present the conditions under which they are global.

The definitions of stability and asymptotic stability used above rely on the norm of a solution being either bounded in any neighborhood of the trivial solution, or decreasing uniformly as  $t$  increases. Particularly for nonlinear systems, these conditions may be difficult to verify. Lyapunov recognized this, and in his 1892 dissertation he suggested that a function other than a norm could be used to determine stability (Lyapunov, 1907). Of course we now refer to such functions as Lyapunov functions, and while they are frequently difficult to find, they offer another powerful tool for classifying equilibria.

Before introducing Lyapunov functions, we define the derivative of a function along a trajectory, in this case where the trajectory is a solution to the initial value problem. For a solution  $x(t)$  to equation A.1 and a real-valued function  $V(t, x)$ , we define (using the terminology of Wiggins, 1996) the *orbital derivative* of  $V$  along  $x$  as



$\dot{V}(t, x) = \limsup_{h \rightarrow 0^+} (1/h)[V(t+h, x+hf(t, x)) - V(t, x)]$ . We now present a version of

Lyapunov's theorem from Hirsch and Smale (1974) for autonomous systems. A function  $V$  satisfying the hypothesis of the theorem is called a *Lyapunov function*.

**Theorem A.4.** (Lyapunov, 1907). Let the autonomous system (equation A.2)  $x'(t) = f(x)$  admit the trivial solution  $f(0) = 0$ . Let  $V(x)$  be a continuous function defined on a neighborhood  $U$  of the origin such that  $0 \in U \subset D$ . Let  $V$  be differentiable on  $U \setminus \{0\}$ , with

$$V(0) = 0, \text{ and } V(x) > 0 \text{ if } x \neq 0. \quad (\text{A.3})$$

Then

$$\dot{V} \leq 0 \text{ on } U \setminus \{0\} \quad (\text{A.4})$$

implies the origin is stable. Furthermore,

$$\dot{V} < 0 \text{ on } U \setminus \{0\} \quad (\text{A.5})$$

implies the origin is asymptotically stable.

**Proof.** Let  $U$  be any neighborhood of the origin, and choose  $\delta$  such that  $B_\delta(0) \subset U$ , where  $B_\delta(0)$  is a  $\delta$ -ball about the origin. Let  $\alpha$  be the minimum value of  $V$  on the boundary of  $B_\delta(0)$ , with  $\alpha > 0$  by equation A.3. Let  $U_1 = \{x \in B_\delta(0) | V(x) < \alpha\}$ . Then no solution starting in  $U_1$  can meet the boundary of  $B_\delta(0)$  since  $V$  is nonincreasing by equation A.4. Therefore, the origin is stable.

If equation A.5 holds as well then  $V$  is strictly decreasing on orbits in  $U \setminus \{0\}$ . Let  $x(t)$  start in  $U_1 \setminus \{0\}$  and suppose  $x(t_n) \rightarrow z_0 \in B_\delta(0)$  for  $t_n \rightarrow \infty$ . (By the compactness of  $B_\delta(0)$  such a sequence exists.) We now show that  $z_0 = 0$ . By the

continuity of  $V$  and equation A.5,  $V(x(t_n)) \rightarrow V(z_0)$  and  $V(x(t)) > V(z_0)$  for all  $t \geq 0$ . If  $z_0 \neq 0$ , let  $z(t)$  be a solution starting at  $z_0$ . For any  $s > 0$ ,  $V(z(s)) < V(z_0)$ , so for  $y(s)$  starting sufficiently near  $z_0$ ,  $V(y(s)) < V(z_0)$ . If we let  $y(0) = x(t_n)$  for sufficiently large  $n$ , then  $V(x(t_n + s)) < V(z_0)$ , which is a contradiction. Therefore,  $z_0 = 0$  is the only possible limit point of  $\{x(t) | t > 0\}$ , and such a limit exists by the compactness of  $B_\delta(0)$ . This proves the theorem.  $\square$

While Lyapunov functions provide a useful technique for classifying equilibria, there is no known algorithm for finding them. In mechanical systems or electrical circuits, potential or total energy is frequently a candidate, but in general no technique works uniformly. Even after 100 years, finding a Lyapunov function is still more of an art than a science. The following example is from Hirsch and Smale (1974).

**Example A.5.** Consider the system

$$x'(t) = 2yz - 2y$$

$$y'(t) = -xz + x$$

$$z'(t) = -z^3.$$

The origin is clearly an equilibrium, but if we linearize the system near the origin via the

Jacobian matrix then  $J|_0 = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , which has eigenvalues 0 and  $\pm\sqrt{2}i$ . We are

unable to classify the origin via linearization, so we seek a Lyapunov function of the form

$V(x, y, z) = ax^2 + by^2 + cz^2$ . Then

$\dot{V} = 2(axy' + byy' + czz') = 2[2axy(z-1) - bxy(z-1) - cz^4]$ . Since we want  $\dot{V} \leq 0$ , letting  $c = 1$  and  $2a = b$  gives  $\dot{V} = -z^4 \leq 0$ , so  $V = x^2 + 2y^2 + z^2$  is a Lyapunov function, and the origin is stable. However, the Lyapunov function is not strict since  $\dot{V} = 0$  on the  $xy$ -plane, and we cannot determine (with this choice of  $V$ ) if the origin is asymptotically stable.  $\square$

While no known algorithm exists to find Lyapunov functions, they have proven very useful over the past one hundred years in determining the stability of equilibria. See, for example, Arnold (1989), Bailey (1966), Bhatia and Szegö (1970), La Salle and Lefschetz (1961), Lehnigk (1966), and Yoshizawa (1966).

In 1962, Bellman recognized that the requirement that a Lyapunov function be real-valued was too restrictive, and he proposed that a vector-valued function might provide more flexibility. He presented his results for square, linear systems and special cases of nonlinear systems. The following lemma is a special case of Theorem A.3 for linear systems, but we present it in its entirety as it provides necessary and sufficient conditions for the solution of a differential equation to be majorized by a solution of a differential inequality. In this lemma,  $A \equiv [a_{ij}] \in \mathbf{R}^{m \times m}$ . The lemma is from Beckenbach and Bellman (1965).

**Lemma A.6.** Let the system of differential equations

$$dx/dt = Ax, \quad x(0) = x_0$$

have a solution  $x(t)$ . Let  $y(t)$  satisfy the differential inequality  $dy/dt \geq Ay$ ,  $y(0) = x_0$ .

Then  $y(t) \geq x(t)$  for  $t \geq 0$  if and only if  $a_{ij} \geq 0$  for  $i \neq j$ .

**Proof.** The system  $dy/dt = Ax + g(t)$ ,  $y(0) = x_0$  has a solution of the form

$$y(t) = x_0 e^{At} + \int_0^t e^{A(t-s)} g(s) ds, \text{ so we need to know when the elements of } e^{At} \text{ are}$$

nonnegative for  $t \geq 0$ . Since  $e^{At} = I + At + A^2 t^2/2! + A^3 t^3/3! + \dots$ , for small positive  $t$   $e^{At} \approx I + At$  so that  $y(t) \geq x(t)$  implies  $a_{ij} \geq 0$ .

Conversely, assume  $a_{ij} > 0$  (the case  $a_{ij} \geq 0$  follows via a limiting procedure) and let  $e^{At} = (e^{At/N})^N$ . For a fixed  $t$ ,  $e^{At/N} = I + A(t/N) + A^2(t/N)^2/2! + \dots$  is positive (in the sense that all elements are positive for  $N$  sufficiently large). Since the product of positive matrices is positive,  $e^{At}$  is positive if  $a_{ij} > 0$  for  $i \neq j$ . This proves the lemma.  $\square$

The condition that  $a_{ij} \geq 0$  for  $i \neq j$  is precisely the condition of  $f(x) = Ax$  being quasimonotone nondecreasing in Theorem A.3 for a linear function. Also, since a linear function is uniformly Lipschitz continuous, Picard's theorem applies and the maximal and minimal solution are the unique solution  $x(t) = x_0 e^{At}$ .

We now state a stability theorem for vector Lyapunov functions, where we determine the stability of the trivial equilibrium solution to equation A.1.

**Theorem A.7.** (Lakshmikantham, 1965). Let  $g \in C[\mathbf{R}_+ \times \mathbf{R}_+^n, \mathbf{R}^n]$ ,  $g(t, 0) \equiv 0$  for all  $t$ , let  $g(t, u)$  be quasimonotone nondecreasing in  $u$  for each  $t \in \mathbf{R}_+$ , and let

$$u'(t) = g(t, u), u(t_0) = u_0 \geq 0. \quad (\text{A.6})$$

Let  $V \in C[\mathbf{R}_+ \times D, \mathbf{R}_+^n]$  and  $V(t, x)$  be locally Lipschitz in  $x$ , with  $\sum_{i=1}^n V_i(t, x) \rightarrow 0$  as

$\|x\| \rightarrow 0$  for each  $t$ ; and let  $f \in C[\mathbf{R}_+ \times D, \mathbf{R}^m]$ ,  $f(t, 0) \equiv 0$ , and for  $(t, x) \in \mathbf{R}_+ \times D$ ,

$$\dot{V}(t, x) \leq g(t, V(t, x)).$$

Then the stability of the trivial solution  $u = 0$  of equation A.6 implies the stability of the trivial solution  $x = 0$  of equation A.1.  $\square$

In the more general theorem of Lakshmikantham (1965), various types of stability and conditional stability are proven as well. Instead of presenting the proof here, we present instead an example from Lakshmikantham (1974) that demonstrates the usefulness of a vector Lyapunov function over that of a scalar Lyapunov function.

**Example A.8.** Consider the system

$$dx/dt = e^{-t}x + y\sin t - (x^3 + xy^2)\sin^2 t, \quad (\text{A.7a})$$

$$dy/dt = x\sin t + e^{-t}y - (x^2y + y^3)\sin^2 t. \quad (\text{A.7b})$$

We would like to determine the stability of the trivial solution to this system, so we attempt to find a scalar Lyapunov function, trying  $V(x, y) = x^2 + y^2$ . The best bound we can achieve with this choice of  $V$  is  $\dot{V} \leq 2(e^{-t} + |\sin t|)V$ . Clearly the trivial solution of  $du/dt = 2(e^{-t} + |\sin t|)u$  is not stable, so we cannot conclude anything about the stability of the trivial solution of equation A.7 with this particular choice of scalar Lyapunov function.

However, if we choose  $V(x, y) \equiv \begin{bmatrix} V_1(x, y) \\ V_2(x, y) \end{bmatrix} = (1/2) \begin{bmatrix} (x+y)^2 \\ (x-y)^2 \end{bmatrix}$  as a vector

Lyapunov function, then  $\dot{V}_1 \leq 2(e^{-t} + \sin t)V_1$ , and  $\dot{V}_2 \leq 2(e^{-t} - \sin t)V_2$ . Since the trivial solution  $u = 0, w = 0$  to the system

$$du/dt = 2(e^{-t} + \sin t)u$$

$$dw/dt = 2(e^{-t} - \sin t)w$$

is clearly stable, the trivial solution  $x = 0, y = 0$  to the system A.7 is also stable.  $\square$

While we used the quantity  $\sum_{i=1}^n V_i(t, x)$  as a measure, we could have used some other measure such as  $\max_i V_i(t, x)$  or  $Q(V_1(t, x), \dots, V_n(t, x))$  where  $Q: \mathbf{R}_+^n \rightarrow \mathbf{R}_+$  and  $Q(u)$  is monotone nondecreasing in  $u$ .

Vector Lyapunov functions offer a lot of flexibility when they can be found, but their true versatility is apparent in the case of conditional stability; i.e., when the stable manifold of the equilibrium has dimension less than  $m$ . Here we present the most basic results of this theory in terms of conditional stability, which are from Lakshmikantham (1965). For completeness we review some of the terminology.

We wish to determine the stability of the (generally trivial) equilibrium solution to equation A.1,

$$x'(t) = f(t, x), \quad x(t_0) = x_0, \quad t_0 \geq 0, \quad (\text{A.1})$$

where  $x \in \mathbf{R}^m$  and  $f(t, x)$  is defined and continuous in  $\mathbf{R}_+ \times \mathbf{R}^m$ . The trivial solution  $x = 0$ , where  $f(t, 0) \equiv 0$  for all  $t \geq 0$ , is *conditionally equistable* (where  $\delta$  in our definition of stability depends on  $\varepsilon$  and  $t_0$ ) if and only if there exists a manifold  $M_{(m-k)}$  through the origin of dimension  $(m - k)$  such that for each  $\varepsilon > 0$  and each  $t_0 \geq 0$  there exists a positive function  $\delta = \delta(t_0, \varepsilon)$  that is continuous in  $t_0$  for each  $\varepsilon$ , where  $\|x(t_0)\| \leq \delta$  and  $x(t_0) \in M_{(m-k)}$  imply  $\|x(t)\| < \varepsilon$  for all  $t \geq t_0$ .

We consider the *comparison system*

$$r'(t) = g(t, r), \quad r(t_0) = r_0 \geq 0 \quad (\text{A.8})$$



where  $g(t, r)$  is quasimonotone nondecreasing in  $r \in \mathbf{R}^n$ . When  $g$  has this property, the solution  $r(t; t_0, r_0)$  is maximal in the sense of component-wise majorization. We consider solutions of the form

$$r_i(t; t_0, r_0) = 0 \quad (i = 1, 2, \dots, k) \quad (\text{A.9a})$$

$$r_i(t; t_0, r_0) > 0 \quad (i = k+1, \dots, n) \quad (\text{A.9b})$$

where  $m - k$  corresponds to the dimension of the manifold  $M_{(m-k)}$ .

The comparison system A.8 is *equistable* (where  $\delta$  in our definition of stability depends on  $\varepsilon$  and  $t_0$ ) if and only if for all  $\varepsilon > 0$ ,  $t_0 \geq 0$ , there exists a positive function

$\delta = \delta(t_0, \varepsilon)$ , continuous in  $t_0$  for each  $\varepsilon$ , such that  $\sum_{i=k+1}^n r_{0_i} \leq \delta$  implies

$\sum_{i=1}^n r_i(t; t_0, r_0) < \varepsilon$  for  $t \geq t_0$ . Other stability and conditional stability properties are

formulated similarly.

The vector Lyapunov function  $V(t, x) \in C[\mathbf{R}_+ \times \mathbf{R}^m, \mathbf{R}_+^n]$  is locally Lipschitz continuous in  $x$ , and its orbital derivative along the trajectories  $x(t)$  of equation A.1 satisfies

$$\dot{V}(t, x) \leq g(t, V(t, x)) \quad (\text{A.10})$$

where  $x(t)$  is any solution such that  $V(t_0, x_0) \leq r_0$ . By Theorem A.3 this ensures that

$V(t, x(t)) \leq r(t; t_0, r_0)$  for  $t \geq t_0$ .

The stable manifold of the origin,  $M_{(m-k)}$ , is defined by the set of points for which

$$V_i(t, x) \equiv 0 \quad \text{for } i = 1, 2, \dots, k < m. \quad (\text{A.11})$$

The bound for the measure,  $b(r)$  is a continuous function, nondecreasing in  $r$ , such that

$$b(r) > 0 \text{ for } r > 0, \text{ and } b(\|x\|) \leq \sum_{i=1}^n V_i(t, x). \quad (\text{A.12})$$

We might also have the bound satisfy  $b(r) \rightarrow \infty$  as  $r \rightarrow \infty$ , which is required for certain types of stability.

While we have used the component-wise sum of the Lyapunov function as the measure, there are several equivalent choices we could have selected. In terms of this measure, we may require, for example,

$$\sum_{i=1}^n V_i(t, x) \rightarrow 0 \text{ as } \|x\| \rightarrow 0 \text{ for each } t \geq 0. \quad (\text{A.13})$$

We now state the theorem on vector Lyapunov functions and conditional stability.

**Theorem A.9.** (Lakshmikantham, 1965). Let assumptions A.10 through A.13 hold. If the solution A.9 to equation A.8 is conditionally equistable, then so is the equilibrium solution of equation A.1

**Proof.** Let  $\varepsilon > 0$ . If  $\|x\| = \varepsilon$ , then from assumption A.12, we have

$$b(\varepsilon) \leq \sum_{i=1}^n V_i(t, x). \quad (\text{A.14})$$

If the stability property holds, given  $b(\varepsilon) > 0$  and  $t_0 \geq 0$ , there exists a positive function

$\delta = \delta(t_0, \varepsilon)$ , continuous in  $t_0$  for each  $\varepsilon$ , such that

$$\sum_{i=1}^n r_i(t; t_0, r_0) < b(\varepsilon) \text{ for } t \geq t_0, \quad (\text{A.15})$$

provided  $\sum_{i=k+1}^n r_{0_i} \leq \delta$ . Let  $x(t)$  be any solution of equation A.1. Then it follows from

assumption A.10 that

$$\sum_{i=1}^n V_i(t, x(t)) \leq \sum_{i=1}^n r_i(t; t_0, r_0) \text{ for } t \geq t_0 \quad (\text{A.16})$$

whenever

$$\sum_{i=1}^n V_i(t_0, x(t_0)) \leq \sum_{i=1}^n r_{0_i}. \quad (\text{A.17})$$

Now choose  $r_{0_i}$  ( $i = 1, 2, \dots, n$ ) to satisfy

$$r_{0_i} = 0 \text{ for } i = 1, 2, \dots, k \text{ and} \quad (\text{A.18})$$

$$\sum_{i=k+1}^n r_{0_i} \leq \delta. \quad (\text{A.19})$$

By equation A.17 and since  $V_i(t, x) \geq 0$  for  $i = 1, 2, \dots, n$ , equation A.18 implies that

$x(t_0) \in M_{(m-k)}$  because of assumption A.11. From the monotonic property of  $b(r)$ ,

assumption A.12, and equations A.17 and A.19, we deduce that  $\|x(t_0)\| \leq b^{-1}(\delta) \equiv \delta_1$ .

By assumption A.13, there exists a  $\delta_2 = \delta_2(t_0, \delta)$  such that  $\sup_{\|x(t_0)\| \leq \delta_2} \sum_{i=1}^n V_i(t_0, x(t_0)) \leq \delta$ .

Let  $\delta_3 = \min\{\delta_1, \delta_2\}$ . It then follows from the choices of  $r_{0_i}$  and  $\delta_3$  that  $x(t_0) \in M_{(n-k)}$

and  $\|x(t_0)\| \leq \delta_3$  implies every solution  $x(t)$  satisfies equation A.16. Suppose, if possible,

that a solution  $x(t)$  of equation A.1 satisfying that  $x(t_0) \in M_{(n-k)}$  and  $\|x(t_0)\| \leq \delta_3$  is such

that  $\|x(\tau)\| = \varepsilon$  for some  $\tau > t_0$ . Then from assumption A.12 and equations A.14, A.15,

and A.16 follows the contradiction  $b(\varepsilon) \leq \sum_{i=1}^n V_i(\tau, x(\tau)) \leq \sum_{i=1}^n r_i(\tau; t_0, r_0) < b(\varepsilon)$ ,

which proves the theorem.  $\square$

Vector Lyapunov functions can also be used to determine certain boundedness properties of solutions of equation A.1, and boundedness theorems analogous to the above theorems for stability are given in Lakshmikantham (1965). We now give an example from Lakshmikantham (1965) of the usefulness of a vector Lyapunov function in determining the conditional stability of an equilibrium.

**Example A.10.** Consider the system of differential equations

$$dx/dt = (1 + \cos(t))x + (1 - \cos(t))y + (\cos(t) - 1)z$$

$$dy/dt = (1 - e^{-t})x + (1 + e^{-t})y + (e^{-t} - 1)z$$

$$dz/dt = (\cos(t) - e^{-t})x + (e^{-t} - \cos(t))y + (e^{-t} + \cos(t))z.$$

We select as the Lyapunov function

$$V(t, x, y, z) \equiv \begin{bmatrix} V_1(t, x, y, z) \\ V_2(t, x, y, z) \\ V_3(t, x, y, z) \end{bmatrix} = \begin{bmatrix} (x + y - z)^2 \\ (x - y + z)^2 \\ (-x + y + z)^2 \end{bmatrix}.$$

Since  $\sum_{i=1}^3 V_i = x^2 + y^2 + z^2 + (x - y)^2 + (y - z)^2 + (z - x)^2$ , we may use as the positive,

monotone function  $b(x, y, z)$  the square of the Euclidean norm, since

$b(x, y, z) = x^2 + y^2 + z^2 \leq \sum_{i=1}^3 V_i(t, x, y, z)$ , so condition A.12 is satisfied. Since

$$\dot{V}_1(t, x, y, z) = 2(x + y - z)(dx/dt + dy/dt - dz/dt) = 4(x + y - z)^2$$

$$\dot{V}_2(t, x, y, z) = 2(x - y + z)(dx/dt - dy/dt + dz/dt) = 4\cos(t)(x - y + z)^2$$

$$\dot{V}_3(t, x, y, z) = 2(-x + y + z)(-dx/dt + dy/dt + dz/dt) = 4e^{-t}(-x + y + z)^2,$$

the comparison system is defined by

$$g_1(t, V_1, V_2, V_3) = 4V_1(t, x, y, z)$$

$$g_2(t, V_1, V_2, V_3) = 4\cos(t)V_2(t, x, y, z)$$

$$g_3(t, V_1, V_2, V_3) = 4e^{-t}V_3(t, x, y, z).$$

The comparison system is therefore

$$\begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} (t) \equiv r'(t) = g(t, r) \equiv \begin{bmatrix} g_1(t, r_1, r_2, r_3) \\ g_2(t, r_1, r_2, r_3) \\ g_3(t, r_1, r_2, r_3) \end{bmatrix} = \begin{bmatrix} 4r_1 \\ 4\cos(t)r_2 \\ 4e^{-t}r_3 \end{bmatrix}.$$

Since each  $g_i$  is only a function of  $r_i$ , the function  $g$  is trivially quasimonotone

nondecreasing, and satisfies  $\dot{V} \leq g(t, V)$ .

Since the trivial solution to  $r_1'(t) = g_1(t, r)$  is clearly unstable, we must choose  $k = 1$ , so the initial condition is  $r_{0_1} = 0$ , and  $r_1(t; t_0, 0) = 0$ . This forces the condition  $V_1(t, x, y, z) = 0$ , which defines the two-dimensional manifold  $M_{(3-1)}$  by the condition  $(x + y - z)^2 = 0$  to be the plane  $x + y = z$ . The solution to the comparison system is

$$r_1(t) = 0,$$

$$r_2(t) = r_{0_2} \exp[4(\sin(t) - \sin(t_0))],$$

$$r_3(t) = r_{0_3} \exp[-4(e^{-t} - e^{-t_0})].$$

If  $\varepsilon > 0$ , then for  $\delta = \min\{\varepsilon/(2e^8), \varepsilon/(2\exp(4e^{-t_0}))\}$ ,  $r_{0_2} + r_{0_3} \leq \delta$  implies  $\sum_{i=1}^3 r_i(t) < \varepsilon$

for  $t \geq t_0$ . Since the conditional equistability condition is satisfied, the trivial solution to the original system is equistable if the initial point is on the manifold  $x + y = z$ .  $\square$

The technique of vector Lyapunov functions is a proven technique in stability analysis in many fields, particularly for large scale dynamical systems and interconnected systems. See, for example, Michel and Miller (1977), Siljak (1978), Lakshmikantham, Matrosov, and Sivasundaram (1991), and Vidyasagar (1993).

## B. CONE-VALUED VECTOR LYAPUNOV FUNCTIONS

While vector Lyapunov functions provide a great amount of flexibility over scalar Lyapunov functions, Lakshmikantham (1974) noted that the quasimonotonicity of the comparison system is not a necessary condition for the system to be stable. In particular, for a linear system, a matrix can still be a stability matrix (have all eigenvalues with real part less than zero) without having all the off-diagonal elements nonnegative.

By requiring the vector Lyapunov function to have each  $V_i \geq 0$ , we have restricted it to the cone  $\mathbf{R}_+^n$ . Lakshmikantham and Leela (1977a) investigated the possibility of selecting a cone other than  $\mathbf{R}_+^n$  to overcome this limitation. The following theorems from Lakshmikantham and Leela (1977b) extend some of the previous results to cones other than  $\mathbf{R}_+^n$ . The first is the comparison principle corresponding to Theorem A.3 through the cone  $K$ .

**Theorem B.1.** (Lakshmikantham and Leela, 1977b). Let  $f \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$  be quasimonotone nondecreasing in  $x$  relative to  $K$  for each  $t \in \mathbf{R}_+$ , and let  $[t_0, \infty)$ ,  $t_0 \geq 0$ , be the largest interval of existence for the maximal solution  $r(t; t_0, x_0)$  of equation A.1 relative to  $K$ . Further, let  $p \in C[\mathbf{R}_+, \mathbf{R}^n]$  and  $D_- p(t) \leq_K f(t, p(t))$  for  $t \geq t_0$ . Then



$p(t_0) \underset{K}{\leq} x_0$  implies  $p(t) \underset{K}{\leq} r(t; t_0, x_0)$  for  $t \geq t_0$ .  $\square$

The quasimonotonicity of  $f(t, x)$  in  $x$  relative to the cone  $P$  does not necessarily imply the quasimonotonicity of  $f(t, x)$  in  $x$  relative to the cone  $Q$  when  $P \subset Q$ . However, if  $P \subset Q$ , then the order relations relative to  $P$  do imply the same order relations relative to  $Q$ . We prove these observations in Section D, and from them comes the following corollary.

**Corollary B.2.** Let  $P$  and  $Q$  be two cones in  $\mathbf{R}^n$  such that  $P \subset Q$ . Let the assumptions of Theorem B.2 hold with  $K$  replaced by  $P$ . Then  $p(t_0) \underset{P}{\leq} x_0$  implies  $p(t) \underset{Q}{\leq} r(t; t_0, x_0)$  for  $t \geq t_0$ .  $\square$

We now state the comparison theorems and stability results for cone-valued vector Lyapunov functions from Lakshmikantham and Leela (1977b). We begin with the system of differential equations

$$x'(t) = f(t, x), \quad x(t_0) = x_0, \quad (\text{A.1})$$

where  $f \in C[\mathbf{R}_+ \times D, \mathbf{R}^m]$ . If  $K$  is a cone in  $\mathbf{R}^n$ ,  $n \leq m$ , and the cone-valued vector Lyapunov function  $V \in C[\mathbf{R}_+ \times D, K]$ , we define for  $(t, x) \in \mathbf{R}_+ \times D$ , the orbital Dini derivative as  $D^+V(t, x) = \limsup_{h \rightarrow 0^+} (1/h)[V(t+h, x+hf(t, x)) - V(t, x)]$ . The first result follows.

**Theorem B.3.** (Lakshmikantham and Leela, 1977b). Assume that  $V(t, x)$  satisfies a local Lipschitz condition in  $x$  relative to  $K$  and for  $(t, x) \in \mathbf{R}_+ \times D$ ,

$$D^+V(t, x) \underset{K}{\leq} g(t, V(t, x)), \quad (\text{B.1})$$

where  $g \in C[\mathbf{R}_+ \times K, \mathbf{R}^n]$ , and  $g(t, u)$  is quasimonotone in  $u$  with respect to  $K$  for each  $t \in \mathbf{R}_+$ . If  $r(t; t_0, u_0)$  is the maximal solution of  $u'(t) = g(t, u)$ ,  $u(0) = u_0$  relative to  $K$ , and  $x(t; t_0, x_0)$  is any solution of equation A.1 such that  $V(t_0, x_0) \leq_K u_0$ , then on the common interval of existence

$$V(t, x(t; t_0, x_0)) \leq_K r(t; t_0, u_0). \quad (\text{B.2})$$

**Proof.** Let  $x(t) = x(t; t_0, x_0)$  be any solution as above. Set  $p(t) = V(t, x(t))$ . For small  $h > 0$ , since  $V(t, x)$  is locally Lipschitz in  $x$  with respect to  $K$ , then

$$p(t+h) - p(t) \leq_K L \|x(t+h) - x(t) - hf(t, x(t))\| + V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)).$$

From this and equation B.1 follows the inequality  $D^+p(t) \leq_K g(t, p(t))$ . Applying Theorem B.1 gives the conclusion B.2.  $\square$

The following variant of this theorem offers more flexibility in applications, and its proof follows from Corollary B.2.

**Theorem B.4.** (Lakshmikantham and Leela, 1977b). Let  $P$  and  $Q$  be two cones in  $\mathbf{R}^n$  such that  $P \subset Q$ . Suppose that  $V \in C[\mathbf{R}_+ \times D, Q]$ ,  $V(t, x)$  satisfies a local Lipschitz condition in  $x$  relative to  $P$ , and  $D^+V(t, x) \leq_P g(t, V(t, x))$  for  $(t, x) \in \mathbf{R}_+ \times D$ , where  $g \in C[\mathbf{R}_+ \times D, \mathbf{R}^n]$  and  $g(t, u)$  is quasimonotone nondecreasing in  $u$  relative to  $P$  for each  $t \in \mathbf{R}_+$ . If  $r(t; t_0, u_0)$  is the maximal solution as in Theorem B.3 relative to  $P$  and  $x(t; t_0, x_0)$  is any solution of equation A.1 such that  $V(t_0, x_0) \leq_P u_0$ , then

$$V(t, x(t; t_0, x_0)) \leq_Q r(t; t_0, x_0) \quad (\text{B.3})$$

on the common interval of existence of  $r(t; t_0, u_0)$  and  $x(t; t_0, u_0)$ . When  $Q = \mathbf{R}_+^n$ , relation B.3 implies the componentwise inequalities  $V(t, x(t; t_0, x_0)) \leq r(t; t_0, u_0)$ .  $\square$

We now state the first stability results for cone-valued vector Lyapunov functions. The first theorem extends the result of Theorem A.7 for equistability. Its proof follows from that of Theorem A.9 and Theorem B.3.

**Theorem B.5.** (Lakshmikantham and Leela, 1977b). Let the assumptions of Theorem B.3 hold. Let  $f(t, 0) \equiv 0$  and  $g(t, 0) \equiv 0$ , and assume that for some  $\phi_0 \in K_0^*$  and  $(t, x) \in \mathbf{R}_+ \times D$ ,

$$b(\|x\|) \leq \langle \phi_0, V(t, x) \rangle \leq a(t, \|x\|), \quad (\text{B.4})$$

where  $a \in C[\mathbf{R}_+ \times [0, \rho), \mathbf{R}_+]$ ,  $b \in C[[0, \rho), \mathbf{R}_+]$ ,  $a(t, 0) \equiv 0$ ,  $b(0) \equiv 0$ ,  $a(t, u)$  and  $b(u)$  are increasing in  $u$ , and  $\rho = \sup\{\|x\| \mid x \in D\}$ . Let the trivial solution  $u \equiv 0$  of  $u'(t) = g(t, u)$  be  $\phi_0$ -equistable, that is, given  $\varepsilon > 0$ ,  $t_0 \geq 0$ , there exists a  $\delta = \delta(t_0, \varepsilon) > 0$  such that  $\langle \phi_0, u_0 \rangle < \delta$  implies  $\langle \phi_0, r(t; t_0, u_0) \rangle < \varepsilon$  for  $t \geq t_0$ . Then the trivial solution  $x \equiv 0$  of equation A.1 is equistable.  $\square$

The following theorem again increases our flexibility when employing cone-valued vector Lyapunov functions. The proof follows from those of Theorems A.9 and B.4.

**Theorem B.6.** (Lakshmikantham and Leela, 1977b). Let the assumptions of Theorem B.4 hold, and let  $f(t, 0) \equiv 0$  and  $g(t, 0) \equiv 0$ . Assume equation B.4 is satisfied for some  $\phi_0 \in Q_0^*$  and that the trivial solution  $u \equiv 0$  of  $u'(t) = g(t, u)$  is  $\phi_0$ -equistable, with  $\phi_0 \in Q_0^*$ . Then the trivial solution  $x \equiv 0$  of equation A.1 is equistable.  $\square$

If  $K = \mathbf{R}_+^n$ , then  $\phi_0 = [1, 1, \dots, 1]^T$ , and Theorem B.5 reduces to Theorem A.7.

In this case, condition B.4 becomes  $b(\|x\|) \leq \sum_{i=1}^n V_i(t, x) \leq a(t, \|x\|)$ . We could again use

measures other than the component-wise sum of the vector Lyapunov function. In

Theorem B.6, if  $P \subset Q = \mathbf{R}_+^n$ , we can remove the requirement for quasimonotonicity if

we select an appropriate cone  $P$  depending on the nature of  $g(t, u)$  as shown in the

following example from Lakshmikantham and Leela (1977b).

**Example B.7.** Consider the system

$$u_1'(t) = a_{11}u_1 + a_{12}u_2 \equiv g_1(t, u_1, u_2), \quad u_1(t_0) = u_{01}, \quad (\text{B.5a})$$

$$u_2'(t) = a_{21}u_1 + a_{22}u_2 \equiv g_2(t, u_1, u_2), \quad u_2(t_0) = u_{02}. \quad (\text{B.5b})$$

If  $Q = \mathbf{R}_+^2$ , and  $a_{21}$  and  $a_{12}$  are not nonnegative, then the function  $g(t, u)$  is not quasimonotone nondecreasing in  $u = (u_1, u_2)$  relative to  $Q$ . Hence, the differential inequalities

$$D^+V_1(t, x) \leq g_1(t, V_1(t, x), V_2(t, x)), \quad (\text{B.6a})$$

$$D^+V_2(t, x) \leq g_2(t, V_1(t, x), V_2(t, x)) \quad (\text{B.6b})$$

do not yield the componentwise estimates of  $V(t, x(t))$  in terms of the solution of equation B.5. However, if there exist two numbers  $\alpha$  and  $\beta$  such that  $0 < \beta < \alpha$  and

$$\alpha^2 a_{21} + \alpha a_{22} \geq \alpha a_{11} + a_{12}, \quad (\text{B.7a})$$

$$\beta^2 a_{21} + \beta a_{22} \geq \beta a_{11} + a_{12}, \quad (\text{B.7b})$$

then these conditions hold with no restriction on the nonnegativity of  $a_{21}$  and  $a_{12}$ . Let

the cone  $P \subset Q = \mathbf{R}_+^2$  be defined by  $P = \{u \in \mathbf{R}_+^2 \mid \beta u_2 \leq u_1 \leq \alpha u_2\}$ . The boundaries for

this cone are  $\alpha u_2 = u_1$  and  $\beta u_2 = u_1$ . On the first boundary, let  $\phi = (-1/\alpha, 1)$  so

$\langle (-1/\alpha, 1), (u_1, u_1/\alpha) \rangle = 0$ , and  $\langle (-1/\alpha, 1), (a_{11}u_1 + a_{12}u_1/\alpha, a_{21}u_1 + a_{22}u_1/\alpha) \rangle \geq 0$  for all  $u \neq 0$ . This reduces to condition B.7a, and we similarly obtain condition B.7b. Thus, if the inequalities B.6 are relative to  $P$ , we obtain component-wise estimates on  $V$  as

$$V_i(t, x(t)) \leq r_i(t; t_0, V(t_0, x_0)) \quad (\text{B.8})$$

by Theorem B.4. If  $a_{12}, a_{21} \geq 0$ , then equation B.8 is the one we would obtain through the standard method of vector Lyapunov functions.  $\square$

While it appears to be a nontrivial exercise to construct an appropriate cone for a system of differential equations, this method still carries much merit as it provides a further increase in our flexibility to determine the stability of equilibria.

These are the most basic results from the theory of cone-valued vector Lyapunov functions we require to motivate the problem of finding a nonnegative cone with respect to which a linear differential operator on  $\mathbf{R}^n$  is quasimonotone nondecreasing. Further directions in the theory of vector Lyapunov functions have been explored recently. In particular, using higher derivatives of vector Lyapunov functions, Köksal and Lakshmikantham (1996) showed how to find a particular cone with respect to which a given comparison system (resulting from taking higher derivatives of a decrescent Lyapunov function) is quasimonotone.

Furthermore, there has been much research on nonlinear comparison systems. See, for example, Hatvani (1984), Deimling and Lakshmikantham (1990), Lakshmikantham, Leela, and Ram Mohan Rao (1991), and Lakshmikantham and Papageorgiou (1994). However, we focus on finding cones for vector Lyapunov functions when the comparison system is linear.



## C. THE QUASIMONOTONICITY OF LINEAR DIFFERENTIAL SYSTEMS

The problem of finding a cone in  $\mathbf{R}^n$  with respect to which a given linear operator is quasimonotone nondecreasing (or essentially nonnegative) and many related problems have been addressed in various forms in the literature in recent years. For example, Vandergraft (1968) following Birkoff (1967) gave sufficient conditions for such a cone to exist without further restrictions on the cone  $K$ . However, Heikkilä first addressed this problem for the application of vector Lyapunov functions, with the requirement that the cone be proper, simplicial, and nonnegative. He stated this to be sufficient for the application of vector Lyapunov functions, and in the next section, we justify this claim. In this section, we summarize Heikkilä's results.

Consider a linear mapping  $A \in \mathbf{R}^{n \times n}$  and a cone  $K$  generated by  $n$  vectors  $b_i$  in  $\mathbf{R}^n$  such that

$$K = \{w_1 b_1 + \dots + w_n b_n | w_i \geq 0\}. \quad (\text{C.1})$$

For C.1 to define a proper, simplicial cone in  $\mathbf{R}^n$ , it is necessary and sufficient that the  $b_i$  be linearly independent. The first result gives necessary and sufficient conditions for a linear operator to be quasimonotone with respect to a cone in  $\mathbf{R}^n$ .

**Theorem C.1.** (Heikkilä, 1980). A linear mapping  $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is quasimonotone nondecreasing relative to the cone generated by a basis  $\{b_1, \dots, b_n\}$  of  $\mathbf{R}^n$  if and only if the matrix of  $A$  relative to this basis has all off-diagonal elements nonnegative.

**Proof.** Let  $A = [a_{ij}]$  and  $C = [c_{ij}]$  be the  $n \times n$  matrices of  $A$  relative to the standard basis of  $\mathbf{R}^n$  and the basis  $\{b_1, \dots, b_n\}$  respectively. Then  $C = B^{-1}AB$ , where



$B \in \mathbf{R}^{n \times n}$  has  $b_j$  as its  $j$ th column. Let  $\phi_i^T$  be the  $i$ th row of  $B^{-1}$ , so

$$\langle \phi_i, b_j \rangle = \delta_{ij}, \quad (C.2)$$

and

$$\langle \phi_i, Ab_j \rangle = c_{ij}. \quad (C.3)$$

Equations C.2 and C.3 imply that  $\phi_i \in K_0^*$  for each  $i = 1, \dots, n$ .

Assume  $c_{ij} \geq 0$  for  $i \neq j$ , and let  $u \in \partial K$  be given. Then if  $u = \sum_{j=1}^n w_j b_j$ , at least

one coefficient  $w_i$  must be zero. Then  $\langle \phi_i, u \rangle = \sum_{j=1}^n w_j \langle \phi_i, b_j \rangle = 0$  and

$$\langle \phi_i, Au \rangle = \sum_{j=1}^n w_j \langle \phi_i, Ab_j \rangle = \sum_{j=1}^n w_j c_{ij} \geq 0 \text{ from equations C.2 and C.3. Thus, the}$$

condition  $c_{ij} \geq 0$  for  $i \neq j$  implies the quasimonotonicity of  $A$  relative to  $K$ .

Conversely, let at least one off-diagonal element of  $C$ , say  $c_{ij}$ , is negative. Then by equation C.3,  $\langle \phi_i, Ab_j \rangle < 0$ . Since the mapping  $u \rightarrow \langle \phi_i, Au \rangle$  is continuous, there exists a  $\delta > 0$  such that  $\|u - b_j\| < \delta$  implies

$$\langle \phi_i, Au \rangle < 0. \quad (C.4)$$

In particular, equation C.4 holds for  $u \in \partial K$  given by  $u = \sum_{k=1}^n w_k b_k$ , with  $w_i = 0$ ,

$w_j = 1$ , and for  $k \neq i, j$ ,  $w_k = \delta/(n\|b_k\|)$ . Since all the coefficients  $w_k$ , except  $w_i$ , are positive, then for any  $b \in K_0^*$  for which  $\langle b, u \rangle = 0$ , it follows that  $\langle b, b_k \rangle = 0$  for all  $k \neq i$ . Thus,  $b$  must be of the form  $b = \alpha \phi_i$  for  $\alpha > 0$ , so by equation C.4,

$\langle b, Au \rangle = \alpha \langle \phi_i, Au \rangle < 0$ . This shows  $A$  is not quasimonotone nondecreasing relative to  $K$  unless  $c_{ij} \geq 0$  for  $i \neq j$ , completing the proof.  $\square$

Salzmann (1972) proved the same result for nonnegative matrices, and he presented it in terms of positive operators on simplicial cones. Furthermore, in his discussion of matrices with invariant cones, Vandergraft (1968) proved a similar result. An immediate consequence is the following.

**Corollary C.2.** (Heikkilä, 1980). If a linear mapping  $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$  has  $n$  linearly independent eigenvectors  $b_1, \dots, b_n$ , then  $A$  is quasimonotone nondecreasing relative to the cone generated by these eigenvectors.

**Proof.** In this case, the matrix of  $A$  relative to  $\{b_1, \dots, b_n\}$  is diagonal.  $\square$

We can also express a necessary condition for monotonicity in terms of the eigenvectors of  $A$ . This result follows from the result from Perron (1907) and Frobenius (1912) on the theory of nonnegative matrices, which states that a nonnegative matrix has a nonnegative eigenvector corresponding to a real eigenvalue of greatest modulus (see Theorem II.B.2).

**Corollary C.3.** (Heikkilä, 1980). If a linear mapping  $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is quasimonotone nondecreasing relative to a cone of type C.1, then  $A$  has a non-zero eigenvector in this cone, and hence a real eigenvalue.

**Proof.** If  $A_q = B^{-1}AB$  is quasimonotone nondecreasing where  $B = [b_1, \dots, b_n]$ , choose a positive number  $r$  so large that  $C = A_q + rI$  is nonnegative. Then  $C$  has a nonnegative eigenvector  $x_1 \geq 0$ , so  $A_q$  (and hence  $A$ ) has an eigenvector  $y_1 = Bx_1 \in K(B)$ .  $\square$

We now consider linear differential systems. By the quasimonotonicity of the linear system

$$u'(t) = A(t)u, \quad t \geq 0, \quad (C.5)$$

relative to a cone  $K$  we mean that for each  $t \geq 0$  the linear mapping  $A(t)$  is quasimonotone nondecreasing relative to  $K$ . The next result for cones contained in  $\mathbf{R}_+^n$  follows from Corollaries C.2 and C.3.

**Corollary C.4.** (Heikkilä, 1980). For the quasimonotonicity of the system C.5 relative to some cone in  $\mathbf{R}_+^n$  it is necessary that for each  $t \geq 0$  at least one eigenvector of  $A(t)$  belongs to  $\mathbf{R}_+^n$ , and sufficient that  $A(t)$  has  $n$  linearly independent eigenvectors in  $\mathbf{R}_+^n$ , which do not depend on  $t$ .  $\square$

Given a nonsingular matrix  $B \in \mathbf{R}^{n \times n}$ , the mapping  $u = Bv$  transforms equation C.5 into the equivalent form

$$v'(t) = B^{-1}A(t)Bv, \quad t \geq 0. \quad (C.6)$$

But  $B^{-1}A(t)B$  is the matrix of the linear mapping  $A(t)$  in the basis of  $\mathbf{R}^n$  formed by the column vectors of  $B$ . Moreover, if  $B$  is nonnegative, these vectors belong to  $\mathbf{R}_+^n$ .

Hence, the next result follows from Theorem C.1.

**Theorem C.5.** (Heikkilä, 1980). There exists a cone  $K$  in  $\mathbf{R}_+^n$  generated by  $n$  linearly independent vectors, relative to which the system C.5 is quasimonotone nondecreasing if and only if there exists a nonsingular matrix  $B$  such that the off-diagonal elements of the coefficient matrix of the system C.6 are nonnegative. Moreover, the column vectors of  $B$  generate  $K$ .  $\square$

This special case of considering cones in  $\mathbf{R}_+^n$  lets us apply Theorem B.6 when the vector Lyapunov function satisfies a differential equality.

#### D. STABILITY THROUGH LINEAR COMPARISON SYSTEMS

The results of Heikkilä (1980) presented in Section C address the problem of finding a cone with respect to which a linear comparison system is quasimonotone, and provide a necessary condition and a sufficient condition for such a cone to exist. Köksal and Lakshmikantham (1996) constructed cones for comparison systems generated by taking derivatives of Lyapunov functions, and Köksal and Fausett (1995) extended these results using generalized eigenvectors.

Heikkilä (1980) states (without proof) that for stability results it is sufficient to consider nonnegative cones. This section gives our proof of this result, and shows it is sufficient to consider square, linear comparison systems.

We wish to determine the stability properties of the origin as a solution to

$$x'(t) = f(t, x) \tag{A.1}$$

where  $f \in C[\mathbf{R}_+ \times \mathbf{R}^m \rightarrow \mathbf{R}^m]$  and  $f(t, 0) \equiv 0$ . To do this we seek a vector Lyapunov function  $V(t, x) \in C[\mathbf{R}_+ \times \mathbf{R}^m, K]$  which is locally Lipschitz in  $x$  with respect to a cone  $K \subset \mathbf{R}^n$ , and which satisfies the differential inequality

$$\dot{V} \leq_K g(t, V) \tag{D.1}$$

where  $\dot{V}$  is the orbital derivative of  $V$  along trajectories  $x(t)$ . If  $g(t, u)$  is quasimonotone nondecreasing with respect to  $K$ , the stability of the trivial solution of

$$u'(t) = g(t, u) \quad (\text{D.2})$$

may be used to determine the stability of the origin  $x(t) = 0$  in equation A.1. We require certain conditions on the Lyapunov function  $V$  for this result to hold, for example,

$$b(\|x\|) \leq \sum_{i=1}^n |V_i| \leq a(\|x\|) \quad (\text{D.3})$$

where  $a$  and  $b$  are as in Theorem B.5.

We let the comparison system A.3 be a square, linear system, or  $g(t, u) = A(t)u$  where  $A: \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ , or for all  $t$ ,  $A \in \mathbf{R}^{n \times n}$ . Let  $V \in \mathbf{R}_+^n$  so the quasimonotonicity of  $A(t)$  implies its off-diagonal elements are nonnegative, and the inequality D.1 is component-wise majorization (otherwise we can change bases so  $V \in \mathbf{R}_+^n$ ).

If  $A(t)$  is a stability matrix, but does not possess the required quasimonotonicity properties, we seek a cone  $K \subset \mathbf{R}^n$  other than  $\mathbf{R}_+^n$  for which inequality D.1 holds, relative to which  $A(t)$  is quasimonotone nondecreasing, and which contains  $V(t, x)$  while maintaining its locally Lipschitz property. The following lemma discusses the effect of a linear transformation on an inequality.

**Lemma D.1.** If  $x \leq_{\mathbf{R}_+^n} y$ , then for a nonsingular matrix  $B \in \mathbf{R}^{n \times n}$ ,  $Bx \leq_{K(B)} By$ ,

where  $K(B) \subset \mathbf{R}^n$  is the cone generated by the columns of  $B$ .

**Proof.** Since  $Bx \leq_{K(B)} By \Leftrightarrow By - Bx \in K(B)$  and  $(y - x) \in \mathbf{R}_+^n$ , then from the linearity of  $B$ ,  $B(y - x) \in K(B) = \{u = \sum_{i=1}^n \alpha_i b_i | \alpha_i \geq 0, b_i \text{ is a column of } B\}$ . Letting  $\alpha_i = (y - x)_i \geq 0$  completes the proof.  $\square$

We can transform a differential system  $v'(t) = Av$ , with a nonsingular matrix  $B$ , where  $y = Bv$ , so  $y'(t) = BAB^{-1}y$ . Such a transformation is called a *Lyapunov Transformation* and the stability of the origin in one system implies the stability of the origin in the other. In this case, we say the systems are equivalent in the sense of Lyapunov (see Gantmacher, 1959).

We now show that a simple coordinate transformation cannot be used to construct a cone which satisfies the hypothesis of Theorem B.6 when the original system does not possess the required quasimonotonicity property.

**Proposition D.2.** Assume a vector Lyapunov function satisfies the inequality

$$\dot{V} \underset{\mathbf{R}_+^n}{\leq} A(t)V, \quad (\text{D.4})$$

but that  $A(t)$  is not quasimonotone nondecreasing with respect to  $\mathbf{R}_+^n$ . Then there does not exist a Lyapunov transformation  $Y = B^{-1}V$  for which  $\dot{Y} \underset{K(B^{-1})}{\leq} C(t)Y$  and  $C(t)$  is quasimonotone nondecreasing with respect to  $K(B^{-1})$ .

**Proof.** If such a transformation existed, then with  $C(t) = B^{-1}A(t)B$ , multiplying inequality D.4 by  $B^{-1}$  and applying Lemma D.1 gives

$$\dot{Y} = B^{-1} \dot{V} \underset{K(B^{-1})}{\leq} B^{-1}A(t)V = B^{-1}BC(t)B^{-1}V = C(t)Y.$$

However, in order that  $C(t)$  be quasimonotone nondecreasing with respect to  $K(B^{-1})$ , by Theorem C.1 it is necessary and sufficient that  $BC(t)B^{-1}$  have its off-diagonal elements nonnegative. But  $BC(t)B^{-1} = A(t)$ , which, by assumption, is not quasimonotone nondecreasing with respect to  $\mathbf{R}_+^n$ .  $\square$



Hence, the problem is to find an appropriate cone with respect to which the function  $A(t)$  is quasimonotone nondecreasing, and for which the differential inequality D.1 and all other required properties are preserved. The following case, where the inequality D.1 is really an equality, is frequently found in applications.

If the orbital derivative of a Vector Lyapunov function  $V \in \mathbf{R}_+^n$  satisfies  $\dot{V} = AV$  where  $A$  is a stability matrix, then the following results give conditions under which the hypotheses of Theorem B.6 are satisfied, and the solution  $x(t) = 0$  to equation A.1 is a stable equilibrium.

First, it is evident that if  $\dot{V} = AV$  then  $B \dot{V} \underset{K}{\leq} BAV$  for all nonsingular transformations  $B$  and all cones  $K$ . We also need the following result.

**Lemma D.3.** If  $V(t, x) \in \mathbf{R}_+^n$  is locally Lipschitz in  $x$  with respect to  $\mathbf{R}_+^n$ , then for any nonsingular matrix  $B \in \mathbf{R}^{n \times n}$ ,  $Y = BV$  is locally Lipschitz in  $x$  with respect to the cone  $K(B)$ .

**Proof.** Evidently,  $V(t, x) \in \mathbf{R}_+^n$  implies  $Y = BV \in K(B)$ . Since  $V$  is locally Lipschitz with respect to  $\mathbf{R}_+^n$ , this implies there exists an  $L \in \mathbf{R}_+^n$  such that

$$L\|x - y\| - (V(t, x) - V(t, y)) \in \mathbf{R}_+^n. \text{ Letting } \hat{L} = BL \in K(B) \text{ yields}$$

$$\hat{L}\|x - y\| - (Y(t, x) - Y(t, y)) \in K(B). \quad \square$$

We also require the result that the Lipschitz property is preserved through cone containment.

**Lemma D.4.** If  $V(t, x) \in P$  is locally Lipschitz in  $x$  with respect to the cone

$P \subset \mathbf{R}^n$ , and  $P \subset Q$  where  $Q \subset \mathbf{R}^n$  is a cone, then  $V(t, x)$  is locally Lipschitz in  $x$  with respect to  $Q$ .

**Proof.** If  $V(t, x)$  is locally Lipschitz in  $x$  with respect to  $P$  implies there exists an  $L \in P$  such that  $L\|x - y\| - (V(t, x) - V(t, y)) \in P$ , then  $L \in Q$  and  $L\|x - y\| - (V(t, x) - V(t, y)) \in Q$ .  $\square$

The following result appears as a remark in Lakshmikantham and Leela (1977).

**Lemma D.5.** If  $P$  and  $Q$  are cones in  $\mathbf{R}^n$  with  $P \subset Q$ , then  $x \leq_P y$  implies  $x \leq_Q y$ .

**Proof.**  $x \leq_P y \Rightarrow y - x \in P \Rightarrow y - x \in Q \Rightarrow x \leq_Q y$ .  $\square$

We now demonstrate the well-known result that if  $P$  and  $Q$  are cones in  $\mathbf{R}^n$  with  $P \subset Q$ , that the quasimonotonicity of a mapping  $A \in \mathbf{R}^{n \times n}$  with respect to  $Q$  does not imply the quasimonotonicity of  $A$  with respect to  $P$ . Let  $P = \mathbf{R}_+^2$  and let  $Q$  be the cone generated by the columns of  $B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ , hence  $P \subset Q$ . Let  $A = \begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix}$ , which is not quasimonotone nondecreasing with respect to  $P$ . Since  $B^{-1}AB = \begin{bmatrix} -3 & 1 \\ 5 & -3 \end{bmatrix}$ , then by Theorem C.1  $A$  is quasimonotone nondecreasing with respect to  $Q$ .

Similarly, if  $P \subset Q$  are cones in  $\mathbf{R}^n$ , the quasimonotonicity of a mapping with respect to  $P$  does not imply the quasimonotonicity of the mapping with respect to  $Q$ .

Letting  $A = \begin{bmatrix} -2 & 2 \\ 1 & 3 \end{bmatrix}$  and choosing  $P$  and  $Q$  as above,  $A$  is quasimonotone

nondecreasing with respect to  $P$ , but  $B^{-1}AB = \begin{bmatrix} -8 & 2 \\ -32 & 9 \end{bmatrix}$ , so by Theorem C.1,  $A$  is not quasimonotone nondecreasing with respect to  $Q$ .

Following Heikkilä (1980) and Köksal and Fausett (1995) we present the following result.

**Proposition D.6.** Suppose a Vector Lyapunov function  $V(t, x) \in \mathbf{R}_+^n$  satisfies  $\dot{V} = A(t)V$  where  $A(t)$  is a stability matrix which is not quasimonotone nondecreasing with respect to  $\mathbf{R}_+^n$ . If the generalized left eigenvectors of  $A(t)$  are all contained in  $\mathbf{R}_+^n$ , there exists a transformation  $Y = B^{-1}V$  and a cone  $K = \mathbf{R}_+^n$  such that the hypotheses of Theorem B.6 are satisfied.

**Proof.** The matrix  $A$  has as its Jordan canonical form  $J = B^{-1}AB$ , where the rows of  $B^{-1}$  are the generalized left eigenvectors of  $A$  and the columns of  $B$  are the generalized eigenvectors of  $A$  (see Horn and Johnson, 1991). By assumption,  $B^{-1}$  is a nonnegative matrix, so  $K(B^{-1}) \subset \mathbf{R}_+^n$ . Hence, the transformation  $Y = B^{-1}V$  yields the differential equation  $\dot{Y} = JY$ . Then  $Y \in K(B^{-1})$  is locally Lipschitz in  $x$  with respect to  $K(B^{-1})$  by Lemma D.4, and with respect to  $\mathbf{R}_+^n$  by Lemma D.5. Since  $J$  is quasimonotone nondecreasing with respect to  $\mathbf{R}_+^n$ , and  $\dot{Y} \leq_{\mathbf{R}_+^n} JY$ , we use  $Y$  as the vector Lyapunov function with  $K = \mathbf{R}_+^n$ , satisfying the conditions of Theorem B.7. The only remaining detail is the measure used to bound  $Y$ , so the following remark completes the proof.  $\square$

**Remark D.7.** Since  $P \subset Q$  implies  $Q^* \subset P^*$ , we cannot, in general, find a

$\hat{\phi} \in Q_0^* = \mathbf{R}_+^n \setminus \{0\}$  such that  $\langle \hat{\phi}, B^{-1}V \rangle$  preserves the original measure

$\langle \phi_0, V \rangle = \sum_{i=1}^n V_i$  with  $\phi_0 = [1, \dots, 1]^T$ . This is because  $\hat{\phi} = B\phi_0$  is not generally in

$Q_0^* = \mathbf{R}_+^n \setminus \{0\}$ , since the nonnegativity of  $B^{-1}$  does not imply the nonnegativity of  $B$ .

However, had we used the equivalent measure  $\max_i |V_i| = \max_i V_i$ , then using as the new measure  $\max_i |Y_i|$ , since each  $Y_i$  is a positive linear combination of the  $V_i$ , preserves the result of Theorem B.6 by scaling  $a$  and  $b$  appropriately in equation B.4.  $\square$

While we used the Jordan form to achieve quasimonotonicity with respect to  $\mathbf{R}_+^n$ , this result may be extended to any case where  $S = B^{-1}AB$ ,  $S$  has nonnegative off-diagonal elements, and the columns of  $B^{-1}$  are contained in  $\mathbf{R}_+^n$ . This follows from Theorem C.5 of Heikkilä (1980), who further shows that this construction is always possible in the case  $n = 2$  if  $A$  has one eigenvector in the first quadrant and at most one of  $a_{12}$  and  $a_{21}$  is negative (in this case it is necessary and sufficient that both eigenvectors be nonnegative). This is an application of Corollary C.5 to square systems with differential equalities.

In the case where we do not begin with an equality in equation D.1, this construction is generally not possible. Instead, given  $\dot{V} \leq_{\mathbf{R}_+^n} A(t)V$ , we now must find a cone with respect to which  $A$  is quasimonotone nondecreasing, on which  $V$  is defined and locally Lipschitz in  $x$ , and on which the inequality D.1 still holds. The result of Kōksal and Lakshmikantham (1996) is a special case of this.

We define the following four properties for a linear comparison system for a vector Lyapunov function  $V$  and a cone  $K$ .

(P1)  $V$  is defined on  $K$ ;

(P2)  $V$  is locally Lipschitz with respect to  $K$ ;

(P3)  $\dot{V} \leq_K A(t)V$ ; and

(P4)  $A(t)$  is quasimonotone nondecreasing with respect to  $K$ .

If we find a cone  $K = P$  for which properties (P1) through (P3) hold, then in order to find a cone  $K = Q$  for which properties (P1) through (P4) hold it is necessary that  $P \subseteq Q$ . This is because if a cone  $Q$  containing  $P$  with respect to which  $A(t)$  is quasimonotone nondecreasing exists, then by Lemmas D.4 and D.5, properties (P1) through (P3) hold for  $Q$ , and we can apply Theorem B.6.

This is essentially what was done in Proposition D.2 where the vector Lyapunov function  $Y \in K(B) \subset \mathbb{R}_+^n$  satisfied  $\dot{Y} \leq_{K(B)} JY$ , and  $J$  was quasimonotone nondecreasing with respect to  $\mathbb{R}_+^n$ . Letting  $K(B) = P$  and  $\mathbb{R}_+^n = Q$  satisfies properties (P1) through (P4).

We first state an obvious result for finding such a cone, which may be useful in applications. The proof follows from the above remarks that  $P \subseteq Q$ .

**Proposition D.8.** If  $V \in P$  satisfies properties (P1) through (P3) for  $K = P$ , and if the cone  $K(B)$  generated by the generalized eigenvectors of  $A$  contains  $P$ , then properties (P1) through (P4) hold for  $K(B)$ .  $\square$

This generalizes to any transformation  $S = B^{-1}AB$  where  $S$  has nonnegative off-diagonal elements and  $K(B)$  contains  $P$ . Following Heikkilä (1980) we next show that to find a cone containing  $\mathbf{R}_+^n$  with respect to which  $A$  is quasimonotone nondecreasing, it is necessary and sufficient to find a cone contained in  $\mathbf{R}_+^n$  with respect to which  $A^T$  is quasimonotone nondecreasing. We use Lemma II.A.3 which states that  $K(B) \subseteq \mathbf{R}_+^n$  if and only if  $K(B^{-1}) \supseteq \mathbf{R}_+^n$ . Since it is presumably easier to find a cone contained in  $\mathbf{R}_+^n$  (the matrix of its generators is nonnegative) than one containing  $\mathbf{R}_+^n$  (as we require) this proposition is useful in applications.

**Proposition D.9.** The matrix  $A$  is quasimonotone nondecreasing with respect to a cone  $K(B^{-1}) \supseteq \mathbf{R}_+^n$  if and only if  $A^T$  is quasimonotone nondecreasing with respect to a cone  $K(B^T) \subseteq \mathbf{R}_+^n$ .

**Proof.** Let  $A^T$  be quasimonotone nondecreasing with respect to a cone  $K(B^T) \subseteq \mathbf{R}_+^n$ . Then by Theorem C.1,  $C = (B^T)^{-1}A^TB^T$  has all of its off-diagonal elements nonnegative, as does  $C^T = BAB^{-1}$ . Since  $K(B^T) \subseteq \mathbf{R}_+^n$  implies  $K(B) \subseteq \mathbf{R}_+^n$ , then  $K(B^{-1}) \supseteq \mathbf{R}_+^n$  and  $A$  is quasimonotone nondecreasing with respect to  $K(B^{-1})$ .

Reversing the argument completes the proof.  $\square$

Hence we can apply Corollary C.4 and Theorem C.5 of Heikkilä (1980) to the search for an appropriate cone, as the generalized eigenvectors of  $A^T$ , or the generalized left eigenvectors of  $A$ , being contained in  $\mathbf{R}_+^n$  is a sufficient condition for the existence of a cone for which property (P4) holds.



Because it is the case most frequently encountered in applications, we have assumed  $P = \mathbf{R}_+^n$  to establish properties (P1) through (P4). Evidently, if  $P \subseteq \mathbf{R}_+^n$  the above results still hold; however, if  $P \supset \mathbf{R}_+^n$  then the construction is not as simple.

Since for two cones  $P$  and  $Q$ ,  $P \subseteq Q$  implies  $Q^* \subseteq P^*$ , then if  $P \not\subseteq \mathbf{R}_+^n$ , instead of seeking a set of vectors  $b_i \in \mathbf{R}_+^n$  such that  $(B^T)^{-1}A^TB^T$  has nonnegative off-diagonal elements, where the  $b_i$  are the columns of  $B^T$ , we must find vectors  $b_i \in P^*$  to construct an appropriate cone. We therefore conclude the discussion of square linear comparison systems with the following extension of Proposition D.9.

**Theorem D.10.** Given a vector Lyapunov function  $V$  and a cone  $P \subset \mathbf{R}^m$  for which properties (P1) through (P3) hold, in order to find a cone  $Q \supseteq P$  with respect to which the matrix  $A$  is quasimonotone nondecreasing it is necessary and sufficient to find  $m$  independent vectors  $b_i \in P^*$  such that if  $B^T = [b_1 \dots b_m]$ , then  $C = (B^T)^{-1}A^TB^T$  has nonnegative off-diagonal elements.

**Proof.** If  $b_i \in P^*$  are such that  $C = (B^T)^{-1}A^TB^T$  has nonnegative off-diagonal elements, then so does  $C^T = BAB^{-1}$ , and  $A$  is quasimonotone nondecreasing with respect to  $K(B^{-1})$  by Theorem C.1. Since  $K(B^{-1}) = K(B^T)^*$ , and since  $K(B^T) \subseteq P^*$  implies  $K(B^T)^* \supseteq P$ , then  $K(B^{-1}) \supseteq P$ . Reversing the argument completes the proof.  $\square$

Proposition D.9, where  $P = \mathbf{R}_+^n$  is a special case of the above result, since  $\mathbf{R}_+^n = \mathbf{R}_+^{n*}$ . Finding an invertible matrix whose columns are contained in  $P^*$  is perhaps less difficult than finding one whose columns generate a cone containing  $P$ . The next result follows from Corollaries C.3 and C.4 of Heikkilä (1980).

**Corollary D.11.** For property (P4) to hold relative to some cone  $Q \supseteq P$ , where  $P$ ,  $Q$ ,  $V$ , and  $A(t)$  are as above, it is necessary that at least one eigenvector of  $A^T(t)$  belong to  $P^*$  and sufficient that  $A^T(t)$  has  $n$  eigenvectors in  $P^*$ , which do not depend on  $t$ .  $\square$

To show quasimonotonicity with respect to an appropriate cone, we only show such a cone exists. In an equivalent approach we could use the cone to transform the vector Lyapunov function and the comparison system as we did in Proposition D.2. This technique now works, since the cone used to make the transformation is contained in our original cone, and the inequality is preserved. The following corollary is an extension of Theorem 2.8.3 of Lakshmikantham, Matrosov, and Sivasundaram (1991).

**Corollary D.12.** If a vector Lyapunov function  $V$  exists such that properties (P1) through (P3) hold for some cone  $P$ , and if a matrix  $B$  exists as in Theorem D.10, then the vector Lyapunov function  $Y = (B^T)^{-1}V$  gives the same conclusion about the stability of the trivial solution to Equation A.1.  $\square$

This approach has the advantage of directly yielding the stable manifold of the origin in the case of conditional stability, but in the usual case it involves an additional computational step. Since the ideas are equivalent, we continue the approach of showing such a cone exists.

We continue our analysis of linear comparison systems considering the case where the system is rectangular instead of square. Let a nonnegative vector Lyapunov function  $V(t, x)$  be such that

$$V: \mathbf{R}_+ \times \mathbf{R}^m \rightarrow \mathbf{R}_+^n, \quad (D.5)$$

whose orbital derivative satisfies inequality D.4,  $\dot{V} \leq A(t)V$ . Let  $A(t) \in \mathbf{R}^{n \times n}$  with

$$A = \begin{bmatrix} & \hat{A} & \tilde{A} \\ \dots & \hat{0} & \dots & \tilde{0} \end{bmatrix}, \quad (\text{D.6})$$

where  $\hat{A} \in \mathbf{R}^{p \times p}$ ,  $\tilde{A} \in \mathbf{R}^{p \times q}$ ,  $\hat{0}$  is a  $q \times p$  zero matrix, and  $\tilde{0}$  is a  $q \times q$  zero matrix, with  $p + q = n$ .

Such a comparison system is possible when the orbital derivative satisfies  $\dot{V}_j \leq 0$  for  $p < j \leq n$ . If we have equality in any of these components, or if  $\dot{V}_j = 0$  for some  $j$ , this is a conserved quantity for our original system A.1, and the theory of conservative systems applies to our analysis. While it is necessary that a conserved quantity be non-constant on open sets, a component of a non-trivial vector Lyapunov function certainly satisfies this requirement, so this more powerful theory applies to the problem. Hence, we assume the orbital derivative satisfies inequalities in its last  $q$  components.

This rectangular  $p \times n$  system can be treated as a square system D.4 where  $A$  is given by equation D.6. Since we address such systems in Chapter IV, where we give a complete theory for reducible, square matrices, there is no need to consider the smaller, rectangular system.

We similarly dismiss the case where the matrix  $A$  in equation D.5 is of the form

$$A = \begin{bmatrix} & \hat{A} & \vdots \\ & & \hat{0} \\ & & \vdots \\ \tilde{A} & & \tilde{0} \end{bmatrix},$$

where  $\tilde{A} \in \mathbf{R}^{q \times p}$  and the last  $q$  columns of  $A$  contain zeroes.  $A$  is not a stability matrix since  $\sigma(A) = \sigma(\hat{A}) \cup \{0_1, \dots, 0_q\}$ , and  $A$  may not possess quasimonotonicity properties, even if  $\hat{A}$  does. However, letting  $Y_i = V_i$  for  $1 \leq i \leq p$  yields the square system  $\dot{Y} \leq \hat{A} Y$ , to which we may apply the theory for square comparison systems, regardless of the nature of  $\tilde{A}$ . Hence, we gain nothing by keeping the last  $q$  components of  $V$ , so we may discard them, leaving the imbedded  $p \times p$  square system.

We have now established the setting for the problem of finding a cone with respect to which a linear comparison system is quasimonotone nondecreasing. It is sufficient to consider square linear systems, and considering the transpose of the comparison system, it is sufficient to find a cone contained in the nonnegative orthant. The next two chapters present our solution to this problem.

#### IV. THE QUASIMONOTONICITY OF A SQUARE, LINEAR OPERATOR WITH RESPECT TO A NONNEGATIVE CONE: THE REAL SPECTRUM

For a matrix  $A \in \mathbf{R}^{n \times n}$  we address the problem “when is  $A$  quasimonotone nondecreasing with respect to a cone in  $\mathbf{R}_+^n$ ?” (or equivalently, when is it essentially nonnegative under a nonnegative change of basis?). In Chapter III we motivated this question with the application of determining stability in dynamical systems via the technique of cone-valued vector Lyapunov functions, and in Chapter VI we present further applications of this problem.

Using Perron-Frobenius theory, Heikkilä (1980) showed a necessary condition (the matrix  $A$  has a nonnegative first eigenvector) and a sufficient condition (the matrix  $A$  has all nonnegative eigenvectors) for such a cone to exist (see Corollary III.C.2) where the first eigenvector is the nonnegative eigenvector  $x_1$  associated with the (real) eigenvalue  $\lambda_1$  of greatest real part.

The next two chapters present our solution to this problem. Using constructive techniques, we bring together the necessary and sufficient conditions Heikkilä gave for  $n > 2$ , and we address the complex spectrum for the first time in this setting. This chapter addresses the case where the spectrum of  $A$  is real, or  $\sigma(A) \subset \mathbf{R}$ , and Chapter V addresses the general spectrum.

##### A. MATRICES WITH A POSITIVE FIRST EIGENVECTOR

We begin with the case where the first eigenvector is positive, or  $x_1 > 0$ . We use a sequence of changes of basis  $A_{m+1} = B_m^{-1} A_m B_m$  with  $K(B_m) \subseteq \mathbf{R}_+^n$ , which ensures the

change of basis is always nonnegative. The following lemma is the key to the construction.

**Lemma A.1.** Let  $B = [x, b_2, \dots, b_n] \in \mathbf{R}^{n \times n}$  be nonsingular with  $x \in (\mathbf{R}_+^n)^\circ$  and

$$B^{-1} = \begin{bmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_n^T \end{bmatrix}.$$

Then there exists a unit basis vector  $e_k \in \mathbf{R}^n$  such that the matrix

$$D^{-1} = \begin{bmatrix} e_k^T \\ (\pm)\phi_2^T \\ \vdots \\ (\pm)\phi_n^T \end{bmatrix}$$

has a nonnegative inverse  $D \in \mathbf{R}_+^{n \times n}$ , where  $(\pm)$  indicates an appropriate choice of sign for each  $\phi_i$ .

**Proof.** Since  $x \in (\mathbf{R}_+^n)^\circ$  and  $\langle \phi_i, x \rangle = 0$  for  $i = 2, \dots, n$ , then  $\pm \phi_i \notin \mathbf{R}_+^n$ .

Hence, each  $\phi_i$  has components of both signs. Since each hyperplane  $\phi_i^\perp$  contains  $x \in (\mathbf{R}_+^n)^\circ$ , then each of these hyperplanes intersects at least  $(n - 1)$  of the coordinate hyperplanes  $e_j^\perp$  in  $\mathbf{R}_+^n$ . (This is evident from the mixed signs in  $\phi_i$  for  $i = 2, \dots, n$ .)

Therefore, there are  $(n - 1)$  hyperplanes which each intersect at least  $(n - 1)$  of the  $(e_j)_+^\perp \equiv e_j^\perp \cap \mathbf{R}_+^n$ . By the pigeonhole principle, at least one  $(e_j)_+^\perp$  intersects each  $\phi_j^\perp$ .

Let this unit basis vector be  $e_k$ . Since we require  $\langle \phi_i, d_i \rangle = 1$  where  $d_i$  is a column of  $D$  for  $i = 2, \dots, n$ , to ensure  $d_i \in (e_k)_+^\perp$  we select the appropriate sign for each  $\phi_i$ .

Therefore,  $D^{-1}$  has as its inverse  $D = [d_1, d_2, \dots, d_n]$ , which is nonnegative.  $\square$



We note that the  $k$ th row of  $D$  is  $e_k^T$ , and that  $d_1 = x/x_k$ .

We now construct a cone in  $\mathbf{R}_+^n$  with respect to which a matrix  $A$  is quasimonotone nondecreasing. We begin with the case where  $A$  is diagonalizable, and the first eigenvalue  $\lambda_1$  is simple. In this case we can construct the matrix

$$B^{-1} = \begin{bmatrix} e_k^T \\ \phi_2^T \\ \vdots \\ \phi_n^T \end{bmatrix}, \quad (\text{A.1a})$$

where the  $\phi_j^T$  are left eigenvectors of  $A$  and  $K((B^{-1})^T) \supseteq \mathbf{R}_+^n$  by Lemma A.1. Then

$B = [x_1, b_2, \dots, b_n]$  is nonnegative, and

$$A_1 = B^{-1}AB = \begin{bmatrix} \lambda_1 & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}. \quad (\text{A.2})$$

If any  $\hat{a}_{1j} < 0$  for  $j = 2, \dots, n$ , then we use another change of basis of the form

$$B_1 = \begin{bmatrix} 1 & \xi_2 & \cdots & \xi_n \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \quad (\text{A.1b})$$

where the  $\xi_i$  are all nonnegative, and

$$A_2 = B_1^{-1}A_1B_1 = \begin{bmatrix} \lambda_1 & a_{12} & \cdots & a_{1n} \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}. \quad (\text{A.3})$$

Since  $a_{ij} = \hat{a}_{ij} + \xi_j(\lambda_1 - \lambda_j)$ , and  $\lambda_1 > \lambda_j$ , we can make  $a_{ij}$  nonnegative, so  $A_2$  is quasimonotone nondecreasing.

In the case where  $\lambda_1$  is not a simple eigenvalue, this construction still works, because for  $\lambda_j = \lambda_1$ ,  $\hat{a}_{ij} = 0$  in equation A.2. To show this, since  $\hat{a}_{ij} = e_k^T A b_j$  and  $b_j$  is in the eigenspace of  $\lambda_j = \lambda_1$  with  $[b_j]_k = 0$  (since  $b_j$  is orthogonal to all of the left eigenvectors  $\phi_i$  where  $\lambda_i \neq \lambda_1$ , and to  $e_k^T$ ). Hence  $\hat{a}_{ij} = e_k^T \lambda_1 b_j = 0$ .

When  $A$  is not diagonalizable, Köksal and Fausett (1996) extended Heikkilä's results using generalized eigenvectors. Following this, we use the “almost diagonal” form of  $A$ , or the canonical form of  $A$  with arbitrarily small  $\varepsilon > 0$  on the super-diagonal (see, for example, Horn and Johnson, 1991). Then, for negative  $\hat{a}_{ij}$ ,  $a_{ij} = \hat{a}_{ij} + \xi_j(\lambda_1 - \lambda_j)$  or  $a_{ij} = \hat{a}_{ij} + \xi_j(\lambda_1 - \lambda_j) - \varepsilon \xi_{j-1}$ , so that  $a_{ij} \geq 0$  for arbitrarily small  $\varepsilon$ . Since the only other nonzero off-diagonals of  $A_2$  are  $\varepsilon > 0$ ,  $A_2$  is again quasimonotone.

The results of the previous two paragraphs combine in the case where  $\lambda_1$  is the eigenvalue which makes  $A$  defective. We summarize our results as follows.

**Theorem A.2.** Let  $A \in \mathbf{R}^{n \times n}$  have a real spectrum and let  $a_{ij} < 0$  for some  $i \neq j$ . In order for  $A$  to be quasimonotone nondecreasing with respect to a nonnegative cone, it is necessary that the first eigenvector  $x_1 \geq 0$ , and sufficient that  $x_1 > 0$ .  $\square$

The disadvantage of the above construction is that we must find all of the generalized left eigenvectors of the matrix  $A$  in to construct the matrix  $A_2$  (and the cone  $K(BB_1)$ ). However, since we only need to know whether such a cone exists for stability results, it may be sufficient to compute only the first eigenvector.

Since in the case where  $x_1$  has some negative component we know no cone exists, and when  $x_1$  is positive a cone always exists, the only remaining case for a real spectrum is when  $x_1 \in \partial \mathbf{R}_+^n$ .

## B. REDUCIBLE MATRICES WITH A NONNEGATIVE FIRST EIGENVECTOR

This section addresses the case where  $x_1 \in \partial \mathbf{R}_+^n$ , or when  $x_1 \geq 0$  has one or more components equal to zero. We assume  $x_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$ , where  $0^q$  indicates the zero vector in  $\mathbf{R}^q$ , and  $p + q = n$ . Because  $x_1 \in \partial \mathbf{R}_+^n$  we cannot necessarily apply Lemma A.1 to the left eigenvectors of  $A$  to obtain a nonnegative change of basis.

Since under a (suitably ordered) nonnegative change of basis  $B$ , with  $x_1 \in K(B)$ ,  $x_1 \mapsto B^{-1}x_1 \in \mathbf{R}_+^p \oplus 0^q$ . From this and Theorem II.B.2, it follows that if  $A_1 = B^{-1}AB =$

$\begin{bmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{bmatrix}$  is quasimonotone, then if  $A'_{11}$  is irreducible, it is no larger than  $p \times p$ .

Furthermore, if  $A'_{22}$  is irreducible, then it can be put into the upper triangular form of

equation A.3 via a transformation  $B^{-1} = \begin{bmatrix} I & 0 \\ 0 & B_{22}^{-1} \end{bmatrix}$ , where  $B_{22}^{-1}$  has the form of equation

A.1. Since this transformation does not effect the nonnegativity of  $A'_{12}$ , in this case the quasimonotone matrix  $A_1 = B^{-1}AB$  has a reduced form with a  $q \times p$  zero block. From this follows our first result about reducible matrices, but we first clarify the particular form the reducible matrix must have.

For a reducible matrix  $A$  with a first eigenvector  $x_1 \in \partial \mathbf{R}_+^n$ , when  $A$  is permuted so that  $x_1$  has all of its zeros in its last  $q$  components and  $A$  has a  $q \times p$

lower-left zero block, we define  $A$  as *reduced*. We treat matrices which are not reduced when  $x_1$  is ordered as above with the irreducible case.

**Theorem B.1.** A reduced matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  is quasimonotone with

respect to a nonnegative cone only if  $A_{11}$  and  $A_{22}$  are quasimonotone nondecreasing with respect to nonnegative cones.

**Proof.** The previous remarks address the case where the quasimonotone form

$$B^{-1}AB = \hat{A}_q = \begin{bmatrix} \hat{A}_{11}^q & \hat{A}_{12}^+ \\ 0 & \hat{A}_{22}^q \end{bmatrix} \text{ is reduced. Consider the matrix } A_q = \begin{bmatrix} A_{11}^q & A_{12}^+ & A_{13}^+ \\ 0 & A_{22}^q & A_{23}^+ \\ 0 & 0 & A_{33}^q \end{bmatrix}$$

where  $A_{ii}^q$  are quasimonotone and irreducible, and  $A_{ij}^+ \geq 0$ . Let the diagonal blocks have dimensions  $r \times r$ ,  $s \times s$ , and  $t \times t$  respectively, where  $r < p$ ,  $t < q$ ,  $r + s > p$ , and  $s + t > q$ . Since the largest eigenvalue  $\mu_1$  of  $A_{22}$  is either an eigenvalue of  $A_{22}^q$  or  $A_{33}^q$ , then  $A_q$  has associated with this eigenvalue an eigenvector which is nonnegative in the last  $s + t$  components. Hence,  $A_{22}$  has a nonnegative eigenvector associated with  $\mu_1$ . The case where this eigenvector is positive is the case of  $\hat{A}_q$  above, and when  $A_{22}$  is reduced, we can repeat this argument. Since the result of this theorem is trivial in the case  $p = 1$  (which is the only case required to prove Theorem C.2 in the next section) then the case where  $A_{22}$  is not reduced follows from that theorem.  $\square$

We now construct a cone when  $A$  is reduced, with  $A_{11} \in \mathbf{R}^{p \times p}$  and  $A_{22} \in \mathbf{R}^{q \times q}$ .

Assume that  $\sigma(A_{11}) = \{\lambda_1 > \lambda_2 \geq \dots \geq \lambda_p\}$  and  $\sigma(A_{22}) = \{\mu_1 > \mu_2 \geq \dots \geq \mu_q\}$  where  $\lambda_1 > \mu_1$  and  $A_{22}$  has a positive first eigenvector.

First let  $B_0 = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$ , where  $B_{11}$  and  $B_{22}$  have the form of equation A.1, so

$$A_1 = B_0^{-1}AB_0 = \begin{bmatrix} \lambda_1 & \alpha_2 \cdots & \alpha_p & & & \\ & \ddots & & & \hat{A}_{12} & \\ 0 & & \lambda_p & & & \\ & & & \mu_1 & \beta_2 \cdots & \beta_q \\ & 0 & & & \ddots & \\ & & & 0 & & \mu_q \end{bmatrix}$$

has upper-left and lower-right blocks with the form of equation A.3. While this transformation does not leave  $A_{12}$  unchanged, we can now let  $\hat{A}_{12}$  be arbitrary.

Next let  $B_1 = \begin{bmatrix} I & B_{12} \\ 0 & I \end{bmatrix}$  where  $B_{12} = \begin{bmatrix} \gamma_1 & \cdots & \gamma_q \\ & 0 & \end{bmatrix}$  with  $\gamma_i \geq 0$  and

$A_2 = B_1^{-1}A_1B_1$ . This only changes the first row of  $\hat{A}_{12}$ , where

$\hat{a}_{1,p+1} \mapsto \hat{a}_{1,p+1} + \gamma_1(\lambda_1 - \mu_1) = a_{1,p+1}$ , and  $\hat{a}_{1,p+j} \mapsto \hat{a}_{1,p+j} - \gamma_1\beta_j + \gamma_j(\lambda_1 - \mu_j) = a_{1j}$  for  $j = 2, \dots, q$ . Since  $\lambda_1 > \mu_j$ ,  $a_{1j}$  can be made arbitrarily large.

Now let  $B_2 = \begin{bmatrix} D_{11} & 0 \\ 0 & I \end{bmatrix}$ , where  $D_{11} = \begin{bmatrix} 1 & & 0 \\ -\delta_2 & \ddots & \\ \vdots & & \ddots \\ -\delta_p & 0 & 1 \end{bmatrix}$  with  $\delta_i \geq 0$ , and

$A_3 = B_2^{-1}A_2B_2$ . While  $B_2$  is not nonnegative, we show that the product  $B_0B_1B_2$  still is, but first we examine the effect of this transformation. The off-diagonals in the first row of  $A_2$  are unchanged under  $B_2$ , as are the last  $q$  rows. For  $i = 2, \dots, p, j = 1, \dots, q$ ,  $a_{i,p+j} \mapsto a_{i,p+j} + \delta_i a_{1j}$ , and since  $a_{1j}$  is arbitrarily large,  $a_{i,p+j}$  can be made nonnegative for arbitrarily small  $\delta_i$ . Furthermore, for  $i = 2, \dots, p; j = 2, \dots, p; i \neq j$ ,  $a_{ij} \mapsto \delta_i \alpha_j > 0$ .

Finally, for  $i = 2, \dots, p$ ;  $a_{i1} \mapsto \delta_i[\lambda_1 - \lambda_i - \sum_{k=2}^p \alpha_k \delta_k]$ , and since  $\lambda_1 > \lambda_i$  and the  $\delta_k$  can be made arbitrarily small, then  $a_{i1} \geq 0$  as well. Hence,  $A_3$  is quasimonotone.

We now show that  $B = B_0 B_1 B_2$  is nonnegative. Since  $B = \begin{bmatrix} B_{11} D_{11} & B_{11} B_{12} \\ 0 & B_{22} \end{bmatrix}$ ,

we only consider  $B_{11} D_{11}$ . If we permute  $A_{11}$  so that when constructing  $B_{11}$ ,  $e_k = e_1$  in

Lemma A.1, then  $B_{11} = \begin{bmatrix} 1 & 0 & & \\ x_2 & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ x_p & b_{p2} & \cdots & b_{pp} \end{bmatrix}$ , where  $b_{ij} \geq 0$  and  $x_i > 0$ , since  $A_{11}$  has a

positive first eigenvector. Only the first column of  $B_{11}$  changes under  $B_{11} D_{11}$ , and

$x_i \mapsto x_i - \sum_{j=2}^p \delta_j b_{ij} \geq 0$  since  $x_i > 0$  and the  $\delta_j$  are arbitrarily small. Therefore, in this

case, a nonnegative cone always exists.

We now show we can relax the assumption that  $A_{11}$  and  $A_{22}$  have first eigenvectors which are strictly positive. If  $A_{11}$  and  $A_{22}$  are each quasimonotone nondecreasing with respect to nonnegative cones, then by the above construction, each can be put in a quasimonotone form with arbitrarily small nonzero elements below the main diagonal. In this case, the above construction still works for the matrix  $A$ . We summarize our results.

**Theorem B.2.** For a reduced matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  with real spectrum and

first eigenvector  $x_1 \in (\mathbb{R}_+^p)^\circ \oplus 0^q$  to be quasimonotone nondecreasing with respect to a nonnegative cone, it is necessary that  $A_{11}$  and  $A_{22}$  are quasimonotone nondecreasing



with respect to nonnegative cones, and sufficient that  $\lambda_1 > \mu_1$ , where  $\lambda_1$  and  $\mu_1$  are the first eigenvalues of  $A_{11}$  and  $A_{22}$  respectively.  $\square$

The only reduced matrices which we have not included in this theorem are those where  $\mu_1 = \lambda_1$ . If both  $A_{11}$  and  $A_{22}$  are in upper-triangular quasimonotone form with  $\mu_1 = \lambda_1$ , this construction may or may not be possible, depending on the sub-matrix  $A_{12}$ .

The following example with  $n = 2$  illustrates this. When  $\lambda > \mu$ ,  $A = \begin{bmatrix} \lambda & a_{12} \\ 0 & \mu \end{bmatrix}$

is quasimonotone with respect to  $K(B)$  where  $B = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$  for sufficiently large  $\gamma$ ,

regardless of the sign of  $a_{12}$ . Also, if  $a_{12} \geq 0$ , then  $A$  is quasimonotone with respect to

$K(I)$ , regardless of  $\lambda$  and  $\mu$ . However, if  $\mu \geq \lambda$  and  $a_{12} < 0$ , then  $A$  is not

quasimonotone with respect to any nonnegative cone. (The case  $\mu > \lambda$  violates the

necessary condition of the first eigenvector being nonnegative.) Furthermore, if  $a_{12} \geq 0$ ,  $A$

has two nonnegative generalized eigenvectors, but if  $a_{12} < 0$ ,  $A$  has a generalized

eigenvector  $x_2 \notin \mathbf{R}_+^2$ .

Using the same construction as we did when  $\lambda_1 > \mu_1$ , we formulate the following sufficient condition for a cone to exist, which we do not describe in terms of generalized eigenvectors, but instead in terms of the sub-matrix  $A_{12}$ .

**Corollary B.3.** For a reduced matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  with real spectrum and

first eigenvector  $x_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$  to be quasimonotone nondecreasing with respect to a nonnegative cone, it is necessary that  $A_{11}$  and  $A_{22}$  are each quasimonotone with respect

to nonnegative cones. Furthermore, if  $A_{11}$  and  $A_{22}$  are in upper-triangular quasimonotone form with  $a_{11} = \lambda_1$  and  $a_{p+1,p+1} = \mu_1 = \lambda_1$ , where  $\mu_1$  is a simple eigenvalue of  $A_{22}$ , then it is sufficient that  $a_{i,p+1} \geq 0$  for  $i = 1, \dots, p$ .  $\square$

We can formulate similar sufficient conditions from this construction if  $\mu_1$  is not a simple eigenvalue of  $A_{22}$ . Given the non-uniqueness of generalized eigenvectors, we have not produced a condition in this case that is based on generalized eigenvectors, although in the case where  $n = 2$  the sufficient condition in Corollary B.3 is also necessary, and the requirement for both generalized eigenvectors to be in  $\mathbf{R}_+^2$  is evident.

### C. IRREDUCIBLE MATRICES WITH A NONNEGATIVE FIRST EIGENVECTOR

This section addresses the case where  $x_1 \in \partial\mathbf{R}_+^n$ , but  $A$  is either irreducible or

not reduced. Consider the matrices  $A = \begin{bmatrix} 7 & -2 & -7 \\ 4 & 1 & -9 \\ -1 & .1 & 1 \end{bmatrix}$  and  $\hat{A} = \begin{bmatrix} 7 & -2 & -7 \\ 6 & -1 & -9 \\ -1 & .1 & 3 \end{bmatrix}$ , which

have  $\lambda_1 = 5$ ,  $\lambda_2 \approx 2.894$ ,  $\lambda_3 \approx 1.106$ , and  $x_1 = [1, 1, 0]^T$ .  $A$  is quasimonotone

nondecreasing with respect to  $K(B)$  where  $B = \begin{bmatrix} 1 & 2 & 15 \\ 1 & 3 & 17 \\ 0 & .05 & 1 \end{bmatrix}$ , yet  $\hat{A}$  is not

quasimonotone nondecreasing with respect to any nonnegative cone.

From the discussion preceeding Theorem B.1, if  $A$  has a quasimonotone form via a nonnegative change of basis, then some such matrix  $A_q$  is reduced with a lower-left zero

block. Furthermore, if  $A_q = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  and  $A_{11}$  has a positive first eigenvector (for

example, if it is irreducible) then we can put  $A_{11}$  in the form of equation A.3 using a transformation of the form of equation A.1. If  $\lambda_1$  is simple, we can use the techniques of Section B for reduced matrices to ensure  $A_{12}$  is nonnegative. This proves the following.

**Lemma C.1.** If a matrix  $A$  with real spectrum and first eigenvector  $x_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$  is quasimonotone nondecreasing with respect to a nonnegative cone, then it is quasimonotone nondecreasing with respect to a nonnegative cone for which  $x_1$  is an extremal.  $\square$

We cannot generally construct a nonnegative cone with respect to which  $A$  is quasimonotone as we did in Sections A and B (since the example in this section shows that such a cone does not always exist) but we can reduce the dimension of the problem by

one by deflating the matrix with the first eigenvector as follows. Let  $B_0 = \begin{bmatrix} (1) & 0 \\ x_1 & I \end{bmatrix}$

where  $I$  is the identity matrix of dimension  $n - 1$ , and  $x_1 = [1, \xi_2, \dots, \xi_p, 0, \dots, 0]^T$ , so

$$A_1 = B_0^{-1}AB_0 = \begin{bmatrix} \lambda_1 & a_{1j} \\ 0 & \hat{a}_{ij} \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 & a_{1j} \\ 0 & A_{22} \end{bmatrix}, \text{ where } \hat{a}_{ij} = a_{ij} - a_{1j}\xi_i, \text{ and for } i > p, \hat{a}_{ij} = a_{ij}.$$

Since Theorem B.1 is trivial in the case  $p = 1$ ,  $A_1$  is quasimonotone with respect to a nonnegative cone only if  $A_{22}$  is, so this is a necessary condition for  $A$  to be quasimonotone with respect to a nonnegative cone. Furthermore, since  $\lambda_1 \geq \lambda_i$  for  $i = 2, \dots, n$  (i.e.,  $\lambda_1$  is an eigenvalue of  $A_{22}$ ) then using a transformation  $B_1$  of the form of equation A.1b, we can make  $A_2 = B_1^{-1}A_1B_1$  quasimonotone as well.

The cone  $K(B_0)$  is a subcone of  $\mathbf{R}_+^p \oplus \mathbf{R}_+^q$ , where we replaced one of the first  $p$  basis vectors with  $x_1$ . The transformation we showed replaced  $e_1$ , but the choice of

which basis vector to replace can be varied by permuting the first  $p$  components of  $x_1$ . It is only necessary that one such transformation produce a matrix  $A_{22}$  which is quasimonotone with respect to a nonnegative cone in order for the above construction to produce a cone for the matrix  $A$ . We summarize these results as follows.

**Theorem C.2.** Let  $A \in \mathbf{R}^{n \times n}$  have a real spectrum with a first eigenvector  $x_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$ , and let  $A$  be irreducible (or not reduced). Then for  $A$  to be quasimonotone nondecreasing with respect to a nonnegative cone it is necessary that for  $B_0 = \begin{bmatrix} (1) & 0 \\ \hat{x}_1 & I \end{bmatrix}$  for some permutation  $\hat{x}_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$  of  $x_1$ , and for  $A_1 = B_0^{-1}AB_0 = \begin{bmatrix} \lambda_1 & a_{1j} \\ 0 & A_{22} \end{bmatrix}$ , that the matrix  $A_{22}$  be quasimonotone nondecreasing with respect to a nonnegative cone. In this case, it is sufficient that  $\lambda_1$  be a simple eigenvalue of  $A$ .  $\square$

When  $\lambda_1$  is not simple the problem is the same as the one discussed in the reduced case, where, for example,  $A = \begin{bmatrix} \lambda & a_{12} \\ 0 & \lambda \end{bmatrix}$ , and the quasimonotonicity depends on the sign of  $a_{12}$ .

The choice of which permutation of  $x_1$  to select is not obvious, but there are only  $p$  such choices that lead to distinct cones. One solution is to compute the second eigenvector  $x_2$  of  $A$ , associated with  $\lambda_2$ , which is the eigenpair required for  $A_{22}$  to be quasimonotone. Since the last  $n - 1$  components of  $B^{-1}x_2$  are the first eigenvector of  $A_{22}$ , then finding a  $B^{-1}$  such that  $B^{-1}x_2 \geq 0$  is necessary, and that the last  $n - 1$  components of  $B^{-1}x_2 > 0$  is sufficient, to determine that such a cone exists.

## V. THE QUASIMONOTONICITY OF A SQUARE, LINEAR OPERATOR WITH RESPECT TO A NONNEGATIVE CONE: THE GENERAL SPECTRUM

### A. IRREDUCIBLE, ESSENTIALLY NONNEGATIVE MATRICES.

In this chapter we discuss the problem of finding a cone with respect to which the matrix  $A$  is quasimonotone nondecreasing in the case where  $A$  has a general (complex) spectrum, or when  $\sigma(A)$  is not strictly real. Because  $A$  is real, its complex eigenvalues and eigenvectors occur in conjugate pairs; however, unlike with the real spectrum, the real-diagonal form of  $A$  is not quasimonotone (it has diagonal blocks which necessarily have a negative off-diagonal element) so we can no longer use the diagonal form.

Additionally, a matrix  $A$  with a general spectrum is not always similar to a quasimonotone matrix, much less through a nonnegative change of basis. The problem of determining when a given set of complex numbers is the spectrum of a nonnegative matrix is known as the nonnegative inverse eigenvalue problem, and we discuss it in the next section. Since a matrix with a real spectrum is always similar to a quasimonotone matrix (its diagonal, or Jordan canonical form) this was not an issue in Chapter IV, and the only task was to produce a nonnegative change of basis. However, in this chapter, the nonnegative inverse eigenvalue problem plays a significant role in our solution.

We have some amount of flexibility over the nonnegative inverse eigenvalue problem in that the matrices we seek are essentially nonnegative, and we can make them nonnegative by shifting the diagonal (and the spectrum) by a positive quantity  $r$ . Since  $B^{-1}AB$  is essentially nonnegative if and only if  $B^{-1}AB + rI$  is nonnegative for sufficiently



large  $r$ , and since  $B^{-1}(A + rI)B = B^{-1}AB + rI$ , for this problem to have a solution the nonnegative inverse eigenvalue problem must have a solution for the set  $\sigma(A) + r$ .

We now show that the matrices we need to consider have an irreducible, essentially nonnegative form. Otherwise, if  $B^{-1}AB = A_q = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  is essentially nonnegative, then the shifted spectra  $\sigma(A_{11}) + r$  and  $\sigma(A_{22}) + r$  must both solve the nonnegative inverse eigenvalue problem. In this case we transform  $A_{12}$  using the construction in Chapter IV, although if  $A_{11}$  and  $A_{22}$  are not reducible, we cannot take advantage of the upper-triangular structure they had in the case of the real spectrum. We use this, and the methods of Chapter IV, to discard a portion of the real part of  $\sigma(A)$ , and we consider only that part which is required to form a solution to the inverse eigenvalue problem. For example, if  $\sigma(A_{22}) \subset \mathbf{R}$ , by using transformations  $B = \begin{bmatrix} I & 0 \\ 0 & B_{22} \end{bmatrix}$  where  $B_{22}$  is formed using the techniques of Chapter IV, we only need the theory for the complex spectrum to treat  $A_{11}$ .

Since the final quasimonotone form is irreducible, it has a positive first eigenvector, so  $A$  having a positive first eigenvector is a necessary condition for this problem to have a solution. (Or as in  $A_q$  above,  $x_1$  must have at least enough positive components to correspond to the dimension of  $A_{11}$ .) Furthermore, from Frobenius (1912) the first eigenvalue  $\lambda_1$  must be simple or the quasimonotone form is reducible.

Since the (shifted) spectrum of  $A$  must solve the nonnegative inverse eigenvalue problem for  $A$  to be quasimonotone nondecreasing with respect to a nonnegative cone,



and it is necessary that the first eigenvector of  $A$  be positive when the quasimonotone form,  $A_q$ , is irreducible. We now show that these conditions are sufficient.

Without loss of generality, assume  $A_q$  is not cyclic (otherwise we can further shift its spectrum so that it is non-cyclic). Let  $D$  be the real-diagonal canonical form of both  $A$  and  $A_q$ , so  $D = B_0^{-1}AB_0 = B_1^{-1}A_qB_1$ . Not only are the spectra of  $A$  and  $A_q$  the same, but the matrices are similar, so that the eigenvalues have the same geometric multiplicity as well. Hence, when we require the shifted spectrum of  $A$  to solve the nonnegative inverse eigenvalue problem, this includes the geometric multiplicity of the eigenvalues. Here,  $B_0 = [x_1, \dots, x_n]$  and  $B_1 = [b_1, \dots, b_n]$  where the  $x_i$  and  $b_i$  are the real parts of the (generalized) eigenvectors of  $A$  and  $A_q$  respectively, and  $x_1$  and  $b_1$  are the positive first eigenvectors associated with the simple eigenvalue  $\lambda_1$  of greatest real part.

While  $A_q = B_1B_0^{-1}AB_0B_1^{-1}$ , it is not generally true that  $B_0B_1^{-1} \geq 0$ , as required.

However, we further change bases using  $A_q^n$  as  $n \rightarrow \infty$  without changing  $A_q$  to make the change of basis nonnegative. Let  $B = B_0B_1^{-1}A_q^n$ , so

$A_q = B^{-1}AB = (A_q^n)^{-1}B_1B_0^{-1}AB_0B_1^{-1}A_q^n$ . Since  $A_q$  is irreducible, non-cyclic, and has a simple first eigenvalue,  $A_q^n \rightarrow [b_1, \dots, b_1]$  as  $n \rightarrow \infty$ . Therefore, as  $n \rightarrow \infty$ ,

$$B \rightarrow B_0B_1^{-1}[b_1, \dots, b_1] = B_0 \begin{bmatrix} 1 & \cdots & 1 \\ & 0 & \end{bmatrix} = [x_1, \dots, x_1] > 0, \text{ so } A \text{ is quasimonotone}$$

nondecreasing with respect to a nonnegative cone.

Hence, given a positive first eigenvector, we have reduced the problem to the nonnegative inverse eigenvalue problem. We summarize our result as follows.

**Theorem A.1.** For a matrix  $A \in \mathbf{R}^{n \times n}$  with complex eigenvalues to be similar to an irreducible, essentially nonnegative matrix by a nonnegative change of basis, it is necessary and sufficient that  $A$  have a positive first eigenvector (associated with the simple real eigenvalue of greatest real part) and that the shifted spectrum of  $A$  solve the nonnegative inverse eigenvalue problem for an irreducible matrix.  $\square$

In the case where the final quasimonotone form is reducible,  $A_{22}$  as well as  $A_{11}$  may have a non-real spectrum, in which case we can apply the above theorem to each of them individually, and then use the techniques of Chapter IV to make  $A_{12}$  nonnegative. In the case where  $\sigma(A_{22})$  is strictly real we can use the techniques of Chapter IV to “throw out” this real part of the spectrum, and construct  $A_{11}$  using the above techniques. The following example shows, however, that we cannot simply “throw out” all but  $\lambda_1$  from the real part of the spectrum.

The matrix  $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$  is a  $4 \times 4$  *circulant* matrix (each row is a shift of

the previous one; see Davis, 1979), which is a type of *cyclic* matrix (its eigenvalues are the four roots of unity). Since  $\sigma(A) = \{\pm 1, \pm i\}$ , this set is a solution to the nonnegative inverse eigenvalue problem. However, we cannot discard the eigenvalue  $\lambda_4 = -1$ , since the set  $\sigma = \{1, i, -i\}$  is not the spectrum of any nonnegative (or quasimonotone) matrix. Furthermore, the left eigenvector associated with  $\lambda_4$  has two components of each sign, so

that the transformation required to throw out  $\lambda_4$ ,  $B^{-1} = \begin{bmatrix} I & 0 \\ \phi_4^T & \end{bmatrix}$  cannot have a nonnegative inverse  $B$ .

When a matrix has a reducible quasimonotone form under a nonnegative change of basis we cannot discard all but the essential real part of the spectrum prior to constructing

$A_q$ . Using transformations of the form  $B^{-1} = \begin{bmatrix} I & 0 \\ \phi_{p+1}^T \\ \vdots \\ \phi_n^T \end{bmatrix}$  for some collection of  $n - p$

left eigenvectors, requires  $B$  to be nonnegative, which is not true in general. However, in the cases where this construction can be done *a priori*, then Theorem A.1 gives sufficient conditions for our problem to have a solution.

To complete this topic, we present a brief discussion of the nonnegative inverse eigenvalue problem.

## B. THE NONNEGATIVE INVERSE EIGENVALUE PROBLEM

There are many types of inverse eigenvalue problems. Given a set of numbers and a structure, they ask when that set of numbers is the spectrum of a matrix with the given structure. For a thorough survey of inverse eigenvalue problems, see Chu (1998). The structure we require is essential nonnegativity, or nonnegativity for the shifted spectrum. The nonnegative inverse eigenvalue problem is one of the classic unsolved problems from the theory of linear algebra (see, for example, Horn and Johnson, 1991) although some

sufficient and some necessary conditions have been given for a set of complex numbers to be the spectrum of a nonnegative matrix.

The nonnegative inverse eigenvalue problem dates back to Perron and Frobenius, and has been approached in numerous forms and variants since then. Necessary conditions for real spectra were given by Suleimanova (1949). His results were improved by Karpelevich (1951), Perfect (1952, 1953, 1955), Suleimanova (1965), and Fiedler (1972). Fiedler (1974) extended these results with the additional requirement that the matrix be symmetric, and Friedland and Karlin (1975) solved a variant of the problem known as the nonnegative additive inverse eigenvalue problem. Most of the results for real spectra are refinements of Suleimanova's necessary condition that the absolute value of the sum of the negative eigenvalues may not exceed  $\lambda_1 > 0$ . Results of this type are always obtainable for essentially nonnegative matrices since we are allowed to shift the spectrum, and as mentioned earlier, the diagonal matrix of any real spectrum is essentially nonnegative.

For complex (not strictly real) spectra, the problem is more difficult. Because of the close relation to nonnegative matrices (see von Mises, 1931) it has been addressed in various forms for nonnegative stochastic matrices, starting with Fréchet (1933), Romanovsky (1936), Dmitriev and Dynkin (1945), Taussky (1948), and Brauer (1952). Brauer presents a summary of the previous results, in addition to his own results and methods of proof. Friedland and Melkman (1979) resolve the question in the case of nonnegative Jacobi matrices, and numerous bounds for the spectra of nonnegative matrices exist (see, for example, Ostrowski and Schneider, 1960) as well as results on the first eigenvector of nonnegative matrices (see, for example, Minc, 1970; or Ashley, 1987),

but these latter results require *a priori* knowledge of the matrix. Additionally, Friedland (1977) extended his previous results for real spectra to complex spectra for the additive and multiplicative nonnegative inverse eigenvalue problem.

Of the more recent results on the nonnegative inverse eigenvalue problem for complex spectra, as it applies to our problem, we mention those which provide sufficient conditions that may be used by the analyst, for example, to draw a stability conclusion in the method of vector Lyapunov functions. Kellogg (1971) addressed the problem of when a set of complex numbers is the spectrum of an essentially positive matrix (hence, for our application, certainly an irreducible essentially nonnegative matrix). He found a sufficient condition to be  $\operatorname{Re}(\lambda_i) - \operatorname{Re}(\lambda_{i+1}) > \sqrt{3} \operatorname{Im}(\lambda_{i+1})$  when the eigenvalues are ordered by their real parts, where  $\lambda_1$  is the first (real) eigenvalue, and the  $\lambda_i$  are the eigenvalues with positive imaginary parts for  $i > 1$ . The other real eigenvalues do not affect this condition. Certainly, the condition is not necessary, as the set  $\{\sqrt{2} + \varepsilon, \pm i\}$  is the spectrum of a positive matrix for all  $\varepsilon > 0$ .

Friedland (1978) addressed the problem for *eventually nonnegative matrices* (where  $A^k$  is nonnegative for all  $k \geq M$  for some  $M > 1$ ) and showed that if  $\sum_{i=1}^n \lambda_i^p \geq 0$  for some  $p \in \{1, 2, \dots\}$  and if  $\lambda_1$  is the only positive  $\lambda_i$  for  $i = 1, \dots, n$ , then  $A$  is eventually nonnegative. This supports the conjecture that if  $k = 1$ , then  $\{\lambda_i\}$  is the spectrum of a nonnegative matrix, although no proof has been found yet. Furthermore, if



$A \geq 0$ ,  $A^k \geq 0$ , it is necessary that  $\sum_{i=1}^n \lambda_i^k \geq 0$ . Similarly, Lowey and London (1978) gave the further necessary condition that  $(\sum_{i=1}^n \lambda_i^k)^m \leq n^{m-1} \sum_{i=1}^n \lambda_i^{km}$  for all  $k, m \in \{1, 2, \dots\}$ .

Boyle, *et. al.*, (1991) characterized via symbolic dynamics the sets of complex numbers which are the nonzero portion of the spectrum of a nonnegative matrix; however, their result is conjecture. Their results are important in that they are presented for *primitive* matrices (those matrices  $A \geq 0$  for which  $A^k > 0$  for some  $k$ ). For the nonnegative inverse eigenvalue problem, this is sufficient as the generalization to irreducible or nonnegative matrices is easily determined. More significant, however, is that for our stability application, since a primitive matrix is a special case of an irreducible nonnegative matrix, this is all we need to apply the techniques of the previous section.

Further results, for example the sufficient conditions of T. J. Laffey or those of Koltracht, Newman, and Xiao (1993) (where Boyle's conjecture is shown to be true for  $n < 5$ ), are discussed in Berman and Plemmons (1994). There are probably enough necessary and sufficient conditions in the literature that most spectra encountered can be determined to be solutions of the nonnegative inverse eigenvalue problem (or not), but as long as the problem is unsolved, the pathological case cannot be discounted.



## VI. FURTHER APPLICATIONS

This chapter presents additional applications that can benefit from the results of this dissertation.

Boyd, *et. al.*, (1994) present the problem of positive orthant stabilizability, which is a special case of the “hit and hold” problem from control theory (see Berman, Neumann, and Stern, 1991). Given a linear differential system  $x' = Ax$  where  $A \in \mathbf{R}^{n \times n}$ , when does  $x(0) \in \mathbf{R}_+^n$  imply  $x(t) \in \mathbf{R}_+^n$  for all  $t \geq 0$ , and  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ ? It is a necessary and sufficient condition that there exist a diagonal matrix  $P > 0$  such that  $PA^T + AP < 0$ , and that  $A$  be essentially nonnegative.

Ohta, Maeda, and Kodama (1984) discuss a similar problem where the state variable is not required to be nonnegative, in which case the problem of finding an essentially nonnegative matrix is the nonnegative inverse eigenvalue problem for the shifted spectrum. They also present the positive realization problem, which is similar to the problem of positive orthant stabilizability, and for which essentially nonnegative matrices play an important role.

In the problem of positive orthant stabilizability, we cannot arbitrarily shift the spectrum of a linear dynamic system without changing the problem. However, if the matrix  $A$  is not essentially nonnegative, but its spectrum satisfies the necessary conditions, then our techniques can find a cone contained in the positive orthant which can be stabilized under the original system. Our techniques can also be used to find a cone containing the positive orthant which can be stabilized under the original system.

A related problem from control theory in Stern (1980) is that of asymptotic holdability, where a trajectory must be held arbitrarily close to a linear subset of  $\mathbf{R}^n$  (not necessarily the nonnegative orthant) for any finite time interval. This leads to positively invariant cones, which Stern (1982) characterized for cones other than  $\mathbf{R}_+^n$ . These are cones  $K$  such that when  $x(t)$  is the solution to  $x' = Ax$ , then  $x(0) \in K$  implies  $x(t) \in K$  for all  $t \geq 0$ . A sufficient condition in this case is the essential nonnegativity of  $A$  with respect to  $K$ .

The hit and hold problem considers a positively invariant simplicial cone  $K$  in  $\mathbf{R}^n$ , and determines the set of initial values  $X_A(K)$  in  $\mathbf{R}^n$  which eventually reach  $K$  (and hence remain there due to the positive invariance of  $K$ ) under  $x' = Ax$ . Neumann and Stern (1985) show that  $X_A(K)$  is itself a cone, and they characterize it for diagonalizable systems  $A$ . Berman, Neumann, and Stern (1986) extend these results to nondiagonalizable systems  $A$ . These results require the spectrum of  $A$  to be real, and a result from Berman, Neumann, and Stern (1991) requires the matrix  $A$  to be essentially nonnegative as a condition of the theorem.

In the case where  $A$  is not essentially nonnegative, the hit and hold problem has not been solved. However, we use our techniques to find a nonnegative cone  $K_1$  with respect to which  $A$  is essentially nonnegative, and the theorem of Berman *et. al.*, can be applied to a smaller cone  $K_1$ . If  $X_A(K_1) \supset K$ , then this is a solution to the hit and hold problem for  $K$ .

These results have applications to economics (see, for example, Sierksma, 1979), engineering, and biology, where frequently state variables are nonnegative and the

essential nonnegativity of the differential system is often required. If a problem is defined on the nonnegative orthant and the operator is not essentially nonnegative, our solutions may be used to find a sub-cone of  $\mathbf{R}^n_+$  for which the problem has the required properties, and to which the results from control theory may be applied.

Bitsoris (1988) studied the hit and hold problem for discrete-time dynamical systems  $x_{k+1} = Ax_k$ , and showed that in addition to invariant cones, invariant polyhedral sets also arise, and these results could be applied to the unsolved problem of determining the spectral properties of systems possessing positively invariant polyhedral cones. For the continuous-time case, Castelan and Hennet (1993) again related the existence of positively invariant polyhedral sets to the problem of finding an essentially nonnegative matrix.

Since we have a technique for finding essentially nonnegative matrices with respect to smaller cones (or larger cones if we consider the transpose of the original system) then depending on the requirements of the application we also have a useful technique for finding positively invariant polyhedral sets in dynamical systems.

The requirement that a change of basis be nonnegative is a natural one for many of these applications, although the techniques we developed can be extended to cones outside of the nonnegative orthant (where for the real spectrum, the problem of seeking an essentially nonnegative matrix becomes trivial, and for the general spectrum is simply the nonnegative inverse eigenvalue problem for the shifted spectrum). Furthermore, our results for complex spectra may be used to extend current results which exist only for matrices with real spectra.

Stern (1982) shows that when the general cone  $K = \mathbf{R}_+^n$ , the problem of positively invariant cones has its solution in the theory of M-matrices. An *M-matrix* (after Minkowski) is a square matrix  $A = rI - B$  where  $B \geq 0$  and  $r \geq \rho(B)$  (or  $r > \rho(B)$  for nonsingular M-matrices). This is a subset of the class of essentially nonpositive matrices with nonnegative diagonals. M-matrices occur frequently in problems from the biological, physical, and social sciences (see, for example, Plemmons, 1977) as well as from the mathematical sciences, and Varga (1976) points out that there is an infrequent exchange of results between these disciplines.

Stern (1981) shows that while a matrix which is cone invariant under the differential system  $x' = Ax$ ,  $x(0) \in K$  is more general than the requirements for  $A$  to be an M-matrix. However, the main properties which hold for M-matrices also hold for  $A$ , which he calls a generalized M-matrix. Numerous classifications of M-matrices have been shown (see, for example, Rothblum, 1979; or Stern and Tsatsumeros, 1987) and most of these are related to the cone invariance of  $A$  or the exponential nonnegativity of  $-A$ .

In terms of other applications of M-matrices, Araki (1975) shows a direct application to the method of vector Lyapunov functions (similar to what we presented in Chapter III), and Kielson and Styan (1973) relate them to the theory of Markov chains which they use to establish results concerning the nonnegative inverse eigenvalue problem for complex spectra.

The negative of an M-matrix is an essentially nonnegative matrix, but the problem we present has a less obvious but more important connection to the theory of M-matrices, although our solution technique does not provide an immediate solution. The inverse M-

matrix problem (see Willoughby, 1977) asks when the inverse of a positive matrix is an M-matrix, and it is also mentioned that nonnegative matrices which are not strictly positive may also have M-matrices as their inverses, but the zeros may only be those associated with reducibility. Since the spectra of M-matrices have certain properties, not every positive matrix solves the inverse M-matrix problem, but as our problem allows a shift in the spectrum, an irreducible solution to our problem may be a way to obtain an M-matrix (or its inverse) which might have use in the numerous applications of M-matrices.

A more general class of matrices than M-matrices is those with nonpositive off-diagonal entries (see Berman, Varga, and Ward, 1978), or essentially nonpositive matrices. This class has an important subclass where all of the principal minors are positive (see Fiedler and Pták, 1962) which is also a more general class than M-matrices. Again, the relationship of these classes of matrices to essentially nonnegative matrices is evident, particularly if we are allowed to shift the spectrum. While the techniques we present do not easily adjust to find a matrix with nonpositive off-diagonal elements, it may be possible to develop a similar technique for essentially nonpositive matrices.

Further classes of matrices studied in relation to the problem of when a matrix is positive on a cone include the *cross-positive matrices*, which are those matrices  $A$  for which  $x \in K$  and  $y \in K^*$  are such that  $\langle x, y \rangle = 0$ , then  $\langle x, Ay \rangle \geq 0$  for a proper, simplicial cone  $K$ . These matrices are discussed in Schneider and Vidyasagar (1970) and Tam (1975), and the former draws the relationship to essentially nonnegative matrices. This relationship is through the exponential of a matrix, where a matrix  $A$  is cross-positive on  $K$  if and only if  $e^A$  is positive on  $K$ . This is true if and only if  $A$  is



essentially nonnegative on  $K$ . While the paper shows the case of  $K$  being the nonnegative orthant, the change of bases we constructed can make the result more general, either in terms of smaller or larger cones. The class of cross-positive matrices includes the *copositive* matrices, which are symmetric and for which  $K = \mathbf{R}_+^n$  (see Haynsworth and Hoffman, 1969). These matrices are also essentially nonnegative, and in addition to the applications previously mentioned, they are important in the field of mathematical programming.

The question of when a linear differential operator is quasimonotone with respect to a nonnegative cone is related to the question of when a linear transformation has an invariant cone. Birkoff (1967) studied this problem and used the results to give a constructive proof of Perron's theorems. Following Birkoff, Vandergraft (1968) characterized the spectral properties of matrices with invariant cones. These results were extended by Barker and Turner (1973) and were related to the nonnegative inverse eigenvalue problem. These ideas have been used in algebraic Perron-Frobenius theory (see Barker and Schneider, 1975) and have been studied geometrically to extend the ideas to non-simplicial cones (see Barker, 1973). An algebraic extension of Lowey and Schneider (1975) further characterizes matrices with invariant cones. These techniques were applied by Vandergraft (1972) to establish the convergence of some splitting methods for solving systems of linear equations. For example, both the Jacobi and Gauss-Seidel methods converge for essentially nonpositive matrices with positive inverses (see Varga, 1962), although to apply our techniques to this problem may be more challenging numerically than solving the original problem  $Ax = b$ .



The question of when a matrix is quasimonotone nondecreasing (or nonincreasing) with respect to a cone is not new, although the requirement that the cone be nonnegative does not appear frequently in recent applications. In applications where the state variables are required to be nonnegative (for example, in population dynamics), and when the original problem is defined on the nonnegative orthant, this requirement is important. These applications, in addition to the one of cone-valued vector Lyapunov functions, demonstrate the usefulness of the results of this dissertation.

1. The first part of the document discusses the importance of maintaining accurate records of all transactions and activities. It emphasizes that proper record-keeping is essential for transparency and accountability, particularly in financial matters. The text suggests that organizations should implement robust systems to track and document every aspect of their operations, from procurement to sales.

2. The second part of the document addresses the challenges of data management in a rapidly changing environment. It highlights the need for flexible and scalable solutions that can adapt to new technologies and evolving business requirements. The author argues that investing in modern data infrastructure is crucial for staying competitive and making informed decisions based on real-time information.

3. The third part of the document explores the role of leadership in driving organizational success. It stresses that effective leaders must inspire and motivate their teams, fostering a culture of innovation and collaboration. The text provides several key strategies for building a high-performing team, including clear communication, delegation, and continuous learning.

4. The fourth part of the document discusses the importance of risk management in achieving long-term goals. It outlines a framework for identifying potential risks, assessing their impact, and implementing mitigation strategies. The author emphasizes that proactive risk management is essential for protecting the organization's assets and ensuring its sustainability.

5. The fifth part of the document focuses on the importance of customer satisfaction and loyalty. It argues that providing exceptional customer service is a key differentiator for businesses in a crowded market. The text offers practical tips for understanding customer needs, addressing complaints, and building lasting relationships with clients.

6. The sixth part of the document discusses the importance of financial planning and budgeting. It stresses that a well-defined financial strategy is essential for managing resources effectively and achieving the organization's financial goals. The author provides guidance on how to develop a realistic budget and monitor financial performance over time.

7. The seventh part of the document explores the importance of innovation and research and development. It argues that investing in new technologies and processes is essential for staying ahead of the competition and creating new opportunities for growth. The text encourages organizations to foster a culture of innovation and to allocate resources to explore new ideas and solutions.

8. The eighth part of the document discusses the importance of sustainability and corporate social responsibility. It argues that businesses have a responsibility to their stakeholders beyond just maximizing profits. The text provides guidance on how to integrate sustainability into the organization's core values and operations, and how to report on its progress.

9. The ninth part of the document discusses the importance of talent management and employee development. It stresses that attracting, retaining, and developing top talent is essential for the organization's long-term success. The text offers strategies for creating a supportive work environment, providing training and development opportunities, and implementing effective performance management systems.

10. The tenth part of the document discusses the importance of strategic planning and execution. It argues that a clear and actionable strategy is essential for guiding the organization's efforts and achieving its vision. The text provides a framework for developing a strategic plan and outlines key steps for implementing it successfully.

## VII. CONCLUSIONS AND FUTURE RESEARCH

This dissertation asks, and answers, the question of when a matrix

$A \in \mathbf{R}^{n \times n}$  is quasimonotone nondecreasing with respect to a nonnegative cone. Chapter III motivates this question with the theory of cone-valued vector Lyapunov functions from stability theory in dynamical systems. Chapter VI presents other applications and related problems where the results can be used. We summarize these results as follows.

A matrix  $A \in \mathbf{R}^{n \times n}$  is quasimonotone nondecreasing with respect to a nonnegative cone if the following conditions hold.

(1) (Heikkilä, 1980). It is necessary that the matrix  $A$  have a real first eigenvalue  $\lambda_1$  such that  $\lambda_1 \geq \operatorname{Re}(\lambda_i)$  for  $i = 2, \dots, n$ . Furthermore, it is necessary that associated with  $\lambda_1$  be a first eigenvector  $x_1 \geq 0$ .

(2) When  $\sigma(A)$ , the spectrum of  $A$ , is real, it is sufficient that  $x_1 > 0$ .

(3) When  $\sigma(A) \subset \mathbf{R}$ ,  $x_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$ , and  $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  is reduced with

$\rho(A_{11}) > \rho(A_{22})$ , it is necessary and sufficient that  $A_{22} \in \mathbf{R}^{q \times q}$  be quasimonotone nondecreasing with respect to a nonnegative cone.

(4) When  $\sigma(A) \subset \mathbf{R}$ ,  $x_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$ , and  $A$  is irreducible (or not reduced) it is necessary that for some permutation  $\hat{x}_1 \in (\mathbf{R}_+^p)^\circ \oplus 0^q$  of  $x_1$ , and for the transformation

$B = \begin{bmatrix} (1) & 0 \\ \hat{x}_1 & I \end{bmatrix}$  with  $A_1 = B^{-1}AB = \begin{bmatrix} \lambda_1 & a_{1j} \\ 0 & A_{22} \end{bmatrix}$ , that the matrix  $A_{22}$  be quasimonotone

nondecreasing with respect to a nonnegative cone. In this case it is sufficient that  $\lambda_1$  be a simple eigenvalue of  $A$ .

(5) When  $\sigma(A) \subset \mathbf{R}$  and  $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$  satisfies the necessary conditions in (3)

above,  $\lambda = \rho(A_{11}) = \rho(A_{22})$ , and  $A_{11}$  and  $A_{22}$  are in upper-triangular quasimonotone nondecreasing form with  $A_{p+1,p+1} = \lambda$ , then it is sufficient that  $A_{i,p+1} \geq 0$  for  $1 \leq i \leq p$ .

(6) For  $\sigma(A) \not\subset \mathbf{R}$ , it is necessary that  $x_1$  have positive components corresponding to  $\lambda_1$  and at least all of the eigenvalues with nonnegative imaginary parts. Furthermore, considering just the irreducible quasimonotone forms (for which  $x_1 > 0$ ) it is necessary and sufficient that the (shifted) spectrum of  $A$  solve the irreducible nonnegative inverse eigenvalue problem.

There are still some unresolved issues for the construction of the cone  $K(B)$ . The case discussed in (5) above (Proposition IV.B.3) is not fully resolved. However, in order to apply the construction in (4) above (the proof of Theorem IV.B.2) to such matrices, the sufficient condition in (5) is also necessary.

In Lemma IV.A.1, constructing the matrix  $B^{-1}$  with nonnegative inverse  $B$ , using equation IV.A.1a requires selecting a basis vector  $e_k$ . Aside from trying all  $n$  possibilities, we do not have an efficient method for selecting  $e_k$ . However, this is only needed when constructing the cone. The application of vector Lyapunov functions does not require the cone's construction, since it is sufficient to know such a cone exists to draw a stability conclusion.

Furthermore, in Section IV.C we replace some  $e_k$ ,  $k \in \{1, \dots, p\}$  with  $x_1$  when constructing the first cone. Again, we may have to try all  $p$  possibilities, although we did present a method of reducing the search. This is more important to applications that only

require the existence of a cone, as this constructive step is required before we can determine the quasimonotonicity of  $A_{22}$ , and hence,  $A$ .

A problem with the above technique, which is certainly a significant problem in applications that require constructing the cone, is that we must compute the generalized eigenvectors of a matrix. In case (4) above, if existence is all we have to determine, then it may be sufficient to compute the first two eigenvectors of  $A$ . However, to find the cone we need to compute all of the generalized left eigenvectors of  $A$ . While theoretically this construction always exists, and while our results give strong existence conclusions, it may be numerically intractable to actually construct the required cone.

Concerning the related problems discussed in Chapter VI, for example the applications of M-matrices, in order to use the techniques of this paper to transform a given matrix into one of a particular form, then the cone  $K(B)$  must certainly be constructed. We do not yet know of a simpler construction than the one we presented.

When the spectrum is complex we have the further problem of determining which part of the real spectrum is required to solve the irreducible nonnegative inverse eigenvalue problem. Certainly we can use (for an irreducible, quasimonotone matrix) no more eigenvalues than there are nonzero components of  $x_1$ , but we could have a further grouping of the complex eigenvalues with different collections of real eigenvalues when the final quasimonotone form of the matrix is reducible. We do not know *a priori* which collections may work, aside from trying all of them. As in all of the above problems, since the method of vector Lyapunov functions is a proven method in examining large scale

dynamical systems (see Siljak, 1978; or Michel and Miller, 1977) we cannot expect to have simple numerical results when we employ these techniques.

Finally, as long as the nonnegative inverse eigenvalue problem is unsolved, then this problem is unsolved for the complex spectrum as well, for we have shown that under the condition of a positive first eigenvector, the problems are equivalent.

These questions present further directions for research, if not in terms of the actual solution to the problem, then at least in terms of the computability and efficiency of the method of solution, particularly when the application requires constructing the cone. We have provided analysts and researchers a basic theory and some useful techniques for approaching these problems.



## LIST OF REFERENCES

- Ando, T., Totally Positive Matrices, *Linear Algebra Applic.*, **90**, 165-219, 1987.
- Ando, T., Majorization, Doubly Stochastic Matrices, and Computation of Eigenvalues, *Linear Algebra Applic.*, **118**, 163-248, 1989.
- Araki, M., Application of M-Matrices to Stability Problems of Composite Dynamical Systems, *J. Math. Anal. Appl.*, **52**, 309-321, 1975.
- Arnold, V. I., *Ordinary Differential Equations* (translated by R. A. Silverman), M.I.T. Press, Cambridge, Massachusetts, 1989.
- Ashley, J., On the Perron-Frobenius Eigenvector for Nonnegative Integral Matrices whose Largest Eigenvalue is Integral, *Linear Algebra Applic.*, **94**, 103-108, 1987.
- Bailey, F. N., The Application of Lyapunov's Second Method to Interconnected Systems, *J. SIAM Control*, **3**, 443-462, 1966.
- Barker, G. P., The Lattice of Faces of a Finite Dimensional Cone, *Linear Algebra Appl.*, **7**, 71-82, 1973.
- Barker, G. P. and Schneider, H., Algebraic Perron-Frobenius Theory, *Linear Algebra Appl.*, **11**, 219-233, 1975.
- Barker, G. P. and Turner, R. E. L., Some Observations on the Spectra of Cone Preserving Maps, *Linear Algebra Applic.*, **6**, 149-153, 1973.
- Beckenbach, E. F. and Bellman, R., *Inequalities*, Springer-Verlag, New York, 1965.
- Bellman, R., Vector Lyapunov Functions, *J. SIAM Control*, **1**, 32-34, 1962.
- Berman, A., *Cones, Matrices, and Mathematical Programming*, Springer-Verlag, New York, 1972.
- Berman, A., Neumann, M., and Stern, R. J., Cone Reachability for Nondiagonal Linear Differential Systems, *Linear Algebra Appl.*, **81**, 263-280, 1986.
- Berman, A., Neumann, M., and Stern, R. J., *Nonnegative matrices in Dynamical Systems*, Wiley-Interscience, New York, 1991.
- Berman, A., and Plemmons, R. J., *Nonnegative Matrices in the Mathematical Sciences*, Society of Industrial and Applied Mathematics, Philadelphia, 1994.

- Berman, A., Varga, R. S., and Ward, R. C., Matrices with Nonpositive Off-Diagonal Entries, *Linear Algebra Appl.*, **21**, 233-244, 1978.
- Bhatia, N. P. and Szegő, G. P., *Dynamical Systems: Stability Theory and Applications*, Springer-Verlag, New York, 1967.
- Bhatia, N. P. and Szegő, G. P., *Stability Theory of Dynamical Systems*, Springer-Verlag, New York, 1970.
- Birkoff, G., Linear Transformations with Invariant Cones, *Amer. Math. Monthly*, **74**, 274-276, 1967.
- Bitsoris, G., Positively Invariant Polyhedral Sets of Discrete-Time Linear Systems, *Int. J. Control*, **47**, 1713-1726, 1988.
- Boyd, S., Ghaoui, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.
- Boyle, M., Handelman, D., Kim, H., Roush, F., The Spectra of Nonnegative Matrices via Symbolic Dynamics, *Annals of Math*, **133**, 249-316, 1991.
- Brauer, A., Limits for the Characteristic Roots of a Matrix IV, *Duke Math J.*, **19**, 75-91, 1952.
- Burton, L. P. and Whyburn, W. M., Minimax Solutions of Ordinary Differential Systems, *Proc. Amer. Math. Soc.*, **3**, 794-803, 1952.
- Bushaw, D., Stability of Lyapunov and Poisson Types, *SIAM Rev.*, **11**, 214-225, 1969.
- Caratheodory, C., Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene werts nicht annahmen, *Math. Ann.*, **64**, 95-115, 1907.
- Castelan, E. B. and Hennes, J. C., On Invariant Polyhedra of Continuous-Time Linear Systems, *IEEE Trans. Automatic Control*, **38**, 1680-1685, 1993.
- Chu, M. T., Inverse Eigenvalue Problems, *SIAM Rev.*, **40**, 1-39, 1998.
- Davis, P. J., *Circulant Matrices*, John Wiley & Sons, New York, 1979.
- Deimling, K. and Lakshmikantham, V., Multivalued Differential Inequalities, *Nonlinear Analysis*, **14**, 1105-1110, 1990.
- Deo, S. G., On Vector Lyapunov Functions, *Proc. Amer. Math. Soc.*, **29**, 575-580, 1971.

- Dmitriev, N. A. and Dynkin, E. B., On the Characteristic Numbers of Stochastic Matrices, *Dokl. Akad. Nauk SSSR*, **49**, 159-162, 1945.
- Elsner, L., Quasimonotonie und Ungleichungen in halbgeordneten Räumen, *Linear Algebra Applic.*, **8**, 249-261, 1974.
- Elsner, L. and Friedland, S., Singular Values, Doubly Stochastic Matrices, and Applications, *Linear Algebra Applic.*, **71**, 77-80, 1985.
- Farkas, J., Über die theorie der einfachen Ungleichungen, *J. Reine Angew. Math.*, **124**, 1-27, 1901.
- Fiedler, M., Bounds for Eigenvalues of Double Stochastic Matrices, *Linear Algebra Applic.*, **5**, 299-310, 1972.
- Fiedler, M., Eigenvalues of Nonnegative Symmetric Matrices, *Linear Algebra Applic.*, **9**, 119-142, 1974.
- Fields, M. and Pták, V., On Matrices with Non-Positive Off-Diagonal Elements and Positive Principal Minors, *Czech. Math. J.*, **12**, 382-400, 1962.
- Fréchet, M., Comportement Asymptotique des Solutions d'un Système d'équations Linéaires et Homogènes aux Différences Finies du Premier Ordre à Coefficients Constants, *Publications de la Faculté des Sciences de l'Université Masaryk (Brno)*, **178**, 1-24, 1933.
- Friedland, S., Inverse Eigenvalue Problems, *Linear Algebra Applic.*, **17**, 15-51, 1977.
- Friedland, S., On an Inverse Problem for Nonnegative and Eventually Nonnegative Matrices, *Israel J. Math.*, **29**, 43-60, 1978.
- Friedland, S., and Karlin, S., Some Inequalities for the Spectral Radius of Nonnegative Matrices and Applications, *Duke Math J.*, **42**, 459-490, 1975.
- Friedland, S., and Melkman, A. A., On the Eigenvalues of Nonnegative Jacobi Matrices, *Linear Algebra Applic.*, **25**, 239-254, 1979.
- Frobenius, G., Über Matrizen aus Positiven Elementen, *S. -B. Deutsch Akad. Wiss Berlin*, 471-476, 1908.
- Frobenius, G., Über Matrizen aus Positiven Elementen II, *S. -B. Deutsch Akad. Wiss Berlin*, 514-518, 1909.
- Frobenius, G., Über Matrizen aus nicht Negativen Elementen, *S. -B. Deutsch Akad. Wiss Berlin*, 456-477, 1912.

Gantmacher, F. R., *The Theory of Matrices*, Vol. 2., Chelsea Publishing Company, New York, 1959.

Goldman, A. J. and Tucker, A. W., Polyhedral Convex Cones, *Linear Inequalities and Related Systems* (H. W. Kuhn and A. W. Tucker Eds.), Princeton University Press, Princeton, 1956.

Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1990.

Habets, P. and Peiffer, K., Classification of Stability-Like Concepts and Their Study Using Vector Lyapunov Functions, *J. Math Anal. Applic.*, **44**, 537-570, 1973.

Hale, J. K. and Koçak, H., *Dynamics and Bifurcations*, Springer-Verlag, New York, 1991.

Hale, J. K. and La Salle, J. P., Differential Equations: Linearity vs. Nonlinearity, *SIAM Rev.*, **5**, 249-272, 1963.

Hartman, P., *Ordinary Differential Equations*, Birkhauser, Boston, 1982.

Hatvani, L., On the Asymptotic Stability by Nondecreasing Ljapunov Function, *Nonlinear Analysis*, **8**, 67-77, 1984.

Haynsworth, E. and Hoffman, A. J., Two Remarks on Copositive Matrices, *Linear Algebra Appl.*, **2**, 387-392, 1969.

Heikkilä, S., On the Quasimonotonicity of Linear Differential Systems, *Applic. Analysis*, **10**, 121-126, 1980.

Heikkilä, S., On the Quasimonotonicity of Differential Systems, *Nonlinear Analysis*, **7**, 91-96, 1983.

Helly, E., Über Mengen konvexer Körper mit gemeinschaftlichen Punkten, *Jahrb. Deut. Math. Verein*, **32**, 175-176, 1923.

Hirsch, M. W. and Smale, S., *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

Horn, R. A. and Johnson, C. R., *Matrix Analysis*, Cambridge University Press, New York, 1991.

Johnson, C. R., Row Stochastic Matrices that are Similar to Doubly Stochastic Matrices, *Linear, Multilin. Algebra*, **10**, 113-120, 1981.



Karlin, S., *Total Positivity*, Stanford University Press, Stanford, 1968.

Karpelevich, F. I., On the Characteristic Roots of Matrices with Nonnegative Elements (Russian), *Isv. Akad. Nauk. SSSR Ser. Mat.*, **15**, 361-383, 1951.

Kellogg, R. B., Matrices Similar to a Poaitive or Essentially Positive Matrix, *Linear Algebra Appl.*, **4**, 191-204, 1971.

Kielson, J. H. and Styan, G. P. H., Markov Chains and M-Matrices--Inequalities and Equalities, *J. Math. Anal. Appl.*, **41**, 439-459, 1973.

Köksal, S. and Fausett, D. W., Variation of Parameters Formula and Lipschitz Stability of Nonlinear Matrix Differential Equations, *Proceedings of the First World Congress of Nonlinear Analysts*, Walter de Gruyter, New York, 1996.

Köksal, S. and Fausett, D. W., *Projections, Cones, and Quasimonotonicity of Linear Differential Systems*, Invited talk at the Second International Conference on Dynamic Systems and Applications, Atlanta, GA, 24-27 May 1995.

Köksal, S. and Lakshmikantham, V., Higher Derivatives of Lyapunov Functions and Cone-Valued Lyapunov Functions, *Nonlinear Analysis*, **26**, 1555-1564, 1996.

Koltracht, I., Newman, M, and Xiao, D., On a Question of Boyle and Handelmann Concerning Eigenvalues of Nonnegative Matrices, *Linear Multilin. Algebra*, **36**, 125-140, 1993.

Krein, M. G. and Rutman, M. A., Linear Operators Leaving Invariant a Cone in a Banach Space, *Uspehi Mat. Nauk (N.S.)*, **3**, 3-95, 1948 [Translation in *Amer. Math. Soc. Translations Ser. 1*, **10**, 199-325, 1956].

Lakshmikantham, V., Vector Lyapunov Functions and Conditional Stability, *J. Math Anal. Applic.*, **10**, 368-377, 1965.

Lakshmikantham, V., On the Method of Vector Lypaunov Functions, *Proceedings of the Twelfth Annual Allerton Conference on Circuit and System Theory*, 71-77, 1974.

Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol. 1., Academic Press, New York, 1969a.

Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol. 2., Academic Press, New York, 1969b.

Lakshmikantham, V. and Leela, S., On the Method of Cone-Valued Lyapunov Functions, *Nonlinear Systems and Applications*, Academic Press, New York, 1977a.

Lakshmikantham, V. and Leela, S., Cone-Valued Lyapunov Functions, *J. Nonlinear Analysis*, **1** 215-222, 1977b.

Lakshmikantham, V., Leela, S., and Ram Mohan Rao, M., New Directions in the Method of Vector Lyapunov Functions, *Nonlinear Analysis*, **16**, 255-262, 1991.

Lakshmikantham, V., Matrosov, V. M., and Sivasundaram, S., *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*, Kluwer Academic Publishers, Boston, 1991.

Lakshmikantham, V. and Papageorgiou, N. S., Cone-Valued Lyapunov Functions and Stability Theory, *Nonlinear Analysis*, **22**, 381-390, 1994.

La Salle, J. and Lefschetz, S., *Stability by Liapunov's Direct Method*, Academic Press, New York, 1961.

Lay, S. R., *Convex Sets and their Applications*, Wiley, New York, 1982.

Lehnigk, S. H., *Stability Theorems for Linear Motion, with an Introduction to Liapunov's Direct Method*, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.

Lindelöf, E., Sur l'application des Méthodes d'approximations Successives à l'étude des Intégrales Réelles des Équations Différentielles Ordinaire, *J. Math. Pures Appl.*, **10**, 117-128, 1894.

Lowey, R. and London, D., A Note On the Inverse Eigenvalue Problem for Nonnegative Matrices, *Linear, Multilin. Algebra.*, **6**, 83-90, 1978.

Lowey, R. and Schneider, H., Indecomposable Cones, *Linear Algebra Appl.*, **11**, 235-245, 1975.

Lyapunov, A. M., Problème Général de la Stabilité du Mouvement, *Ann. Fac. Sci. Univ. Toulouse*, **9**, 203-475, 1907 [Reproduced in *Ann. Math. Study* (17), Princeton, 1947].

Lynn, M. S. and Timlake, W. P., Bounds for Perron Eigenvectors and Subdominant Eigenvalues of Positive Matrices, *Linear Algebra Applic.*, **2**, 143-152, 1969.

Massera, J. L., On Liapounoff's Conditions of Stability, *Ann. of Math.*, **50**, 705-721, 1949.

Massera, J. L., Contributions to Stability Theory, *Ann. of Math.*, **64**, 182-206, 1956.

Michel, A. N., and Miller, R. K., *Qualitative Analysis of Large Scale Dynamical Systems*, Academic Press, New York, 1977.



- Minc, H., On the Maximum Eigenvector of a Positive matrix, *SIAM J. Numer. Anal.*, **7**, 424-427, 1970.
- Minc, H., Linear Transformations of Nonnegative Matrices, *Linear Algebra Applic.*, **9**, 149-153, 1974.
- Mirsky, L., Inequalities and Existence Theorems in the Theory of Matrices, *J. Math. Anal. Appl.*, **9**, 99-118, 1964.
- Mises, R. von, *Vorlesungen aus dem Gebiete der angewandten Mathematic*, Band 1, *Wahrscheinlichkeitsrechnung und ihre Anwendung in der Statistik und theoretischen Physik*, Leipzig, 1931.
- Müller, M., Über das Fundamentaltheorem in der Theorie der Gewöhnlichen Differentialgleichungen, *Math. Z.*, **26**, 619-645, 1926.
- Neumann, M. and Stern, R. J., Cone Reachability for Linear Differential Systems, *Applic. Analysis*, **20**, 57-71, 1985.
- Ohta, Y., Maeda, H., and Kodana, S., Reachability, Observability, and Realizability of Continuous-Time Positive Systems, *SIAM J., Control Optm.*, **22**, 171-180, 1984.
- Ostrowski, A. M. and Schneider, H., Bounds for the Maximal Characteristic Root of a Nonnegative, Irreducible Matrix, *Duke Math J.*, **27**, 547-553, 1960.
- Peano, G., Démonstration de l'intégrabilité des Équations Différentielles Ordinaires, *Math. Ann.*, **37**, 182-228, 1890.
- Perfect, H., On Positive Stochastic Matrices with Real Characteristic Roots, *Proc. Cambridge Phil. Soc.*, **48**, 271-276, 1952.
- Perfect, H., Methods of Constructing Certain Stochastic Matrices, *Duke Math J.*, **20**, 395-404, 1953.
- Perfect, H., Methods of Constructing Certain Stochastic Matrices II, *Duke Math J.*, **22**, 305-311, 1955.
- Perron, O., Zur Theorie der Matrizen, *Math. Ann.*, **64**, 248-263, 1907.
- Picard, É., Mémoire sur la Théorie des Équations aux Dérivées Partielles et la Méthode des Approximations Successives, *J. Math. Pures Appl.*, **6**, 423-441, 1890.
- Plemmons, R. J., M-Matrix Characterizations--Nonsingular M-Matrices, *Linear Algebra Appl.*, **18**, 175-188, 1977.

- Rockafellar, R. T., *Convex Analysis*, Princeton University Press, Princeton, 1970.
- Romanovsky, V., Recherches sur les Chaînes de Markoff, *Acta. Mathematica*, **66**, 147-251, 1936.
- Rothblum, U. G., An Index Classification of M-Matrices, *Linear Algebra Appl.*, **23**, 1-12, 1979.
- Salzmann, F. L., A Note on the Eigenvalues of Nonnegative Matrices, *Linear Algebra Applic.*, **5**, 329-338, 1972.
- Sandgren, L., On Convex Cones, *Math. Scand.*, **2**, 19-28, 1954.
- Schneider, H. and Turner, R. E. L., Positive Eigenvectors of Order-Preserving Maps, *J. Math. Anal. Appl.*, **37**, 506-515, 1972.
- Schneider, H. and Vidyasagar, M., Cross-Positive Matrices, *SIAM J. Numer. Anal.*, **7**, 508-519, 1970.
- Seneta, E., *Nonnegative Matrices*, Springer-Verlag, New York, 1981.
- Sierksma, G., Nonnegative Matrices: The Open Leontief Model, *Linear Algebra Appl.*, **26**, 175-201, 1979.
- Siljak, D. D., *Large Scale Dynamic Systems*, North-Holland, New York, 1978.
- Stern, R. J., Asymptotic Holdability, *IEEE Trans. Automatic Control*, **6**, 1196-1198, 1980.
- Stern, R. J., Generalized M-Matrices, *Linear Algebra Appl.*, **41**, 201-208, 1981.
- Stern, R. J., A Note on Positively Invariant Cones, *Appl. Math. Optim.*, **9**, 67-72, 1982.
- Stern, R. J. and Tsatsomeros, M., Extended M-Matrices and Subtangentiality, *Linear Algebra Appl.*, **97**, 1-11, 1987.
- Stoer, J. and Witzgall, C., *Convexity and Optimization in Finite Dimensions I*, Springer-Verlag, New York, 1970.
- Suleimanova, H. R., Stochastic Matrices with Real Characteristic Values, *Dokl. Akad. Nauk. SSSR*, **56**, 343-345, 1949.

Suleimanova, H. R., The Question of a Necessary and Sufficient Condition for the Existence of a Stochastic Matrix with Prescribed Characteristic Numbers, *Trudy Vsesojuz Zaocn Energet Inst. Vyp.*, **28**, 33-49, 1965.

Tam, B. S., A Note on Cross-Positive Matrices, *Linear Algebra Appl.*, **12**, 7-9, 1975.

Taussky, O., Bounds for Characteristics Roots of Matrices, *Duke Math. J.*, **15**, 1043-1044, 1948.

Vandergraft, J. S., Spectral Properties of Matrices which have Invariant Cones, *SIAM J. Appl. Math.*, **16**, 1208-1222, 1968.

Vandergraft, J. S., Applications of Partial Orderings to the Study of Positive Definiteness, Monotonicity, and Convergence, *SIAM J. Numer. Anal.*, **9**, 97-104, 1972.

Varga, R. S., *Matrix Iterative Analysis*, Prentice-Hall, London, 1962.

Varga, R. S., On Recurring Theorems on Diagonal Dominance, *Linear Algebra Appl.*, **13**, 1-9, 1976.

Verhulst, F., *Nonlinear Differential Equations and Dynamical Systems*, Springer, New York, 1996.

Vidyasagar, M., *Nonlinear Systems Analysis*, Prentice-Hall, New Jersey, 1993.

Walter, W. G., *Differential and Integral Inequalities*, Springer-Verlag, New York, 1970.

Weyl, H., Elementare Theorie der konvexen Polyeder, *Commentarii Mathematici Helvetici*, **7**, 260-306, 1935.

Wiggins, S., *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, New York, 1996.

Willoughby, R. A., The Inverse M-Matrix Problem, *Linear Algebra Appl.*, **18**, 75-94, 1977.

Yoshizawa, T., *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, Tokyo, 1966.



## INITIAL DISTRIBUTION LIST

1. Defense Technical Information Center.....2  
8725 John J. Kingman Rd., STE 0944  
Ft. Belvoir, VA 22060-6218
2. Dudley Knox Library..... 2  
Naval Postgraduate School  
411 Dyer Rd.  
Monterey, CA 93943-5101
3. Professor D. Canright, Code MA/Ca..... 1  
Department of Mathematics  
Naval Postgraduate School  
Monterey, CA 93943
4. Professor C. Frenzen, Code MA/Fr..... 1  
Department of Mathematics  
Naval Postgraduate School  
Monterey, CA 93943
5. Professor R. Dell, Code OR/De..... 1  
Department of Operations Research  
Naval Postgraduate School  
Monterey, CA 93943
6. Professor C. Scandrett, Code MA/Sd..... 1  
Department of Mathematics  
Naval Postgraduate School  
Monterey, CA 93943
7. Professor M. Weir, Code MA/Wc..... 1  
Department of Mathematics  
Naval Postgraduate School  
Monterey, CA 93943
8. Professor W. M. Woods, Code MA/Wo..... 1  
Chairman, Department of Mathematics  
Naval Postgraduate School  
Monterey, CA 93943
9. Professor M. Marcus..... 1  
2937 Kenmore Place  
Santa Barbara, CA 93105

10.	Professor D. Fausett.....	1
	146 Lynwood Dr.	
	Aiken, SC 29803	
11.	Professor V. Lakshmikantham.....	1
	Department of Applied Mathematics	
	150 W. University Blvd.	
	Melbourne, FL 32901	
12.	COL D. C. Arney.....	1
	Department of Mathematical Sciences	
	United States Military Academy	
	West Point, NY 10996	
13.	Philip Beaver.....	5
	P. O. Box 1230	
	Santa Barbara, CA 93102	





12 483NP6 3177  
TH  
10/99 22527-200 FILE









DUDLEY KNOX LIBRARY



3 2768 00368313 7